W3203
Discrete Mathematics

Logic and Proofs

Spring 2015
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Outline

- Propositional Logic
- Operators
- Truth Tables
- Logical Equivalences
- Laws of Logic
- Rules of Inference
- Quantifiers
- Proof Patterns
- Text: Rosen 1
- Text: Lehman 1-3
Logic Puzzle

- Three kinds of people live on an island:
  - *Knights (K)*: always tell the truth
  - *Knaves (V)*: always lie
  - *Spies (S)*: either lie or tell the truth

- You meet 3 people, A, B, and C
  - You know one is K, one is V, and one is S
  - Each of them knows all of their types
  - They make three statements about each other
  - Can you determine who is the knight/knave/spy?
Logic Puzzle

- Statements:
  - $A$: “I am the knight”
  - $B$: “A is not the knave”
  - $C$: “B is not the knave”

- Can you determine who is the knight/knave/spy?
Logic Puzzle (solution)

- Statements:
  - A: “I am the knight”
  - B: “A is not the knave”
  - C: “B is not the knave”

- Can you determine who is the knight/knave/spy?
  - Suppose A is the Knight (K). Then B tells truth, B must be a spy (S). But C tells truth, can’t be a knave (V).
  - Suppose B is K. Then B tells truth, A must be S. Hence C is V, but he tells truth. Hence we have a contradiction.
  - C must be K. Then C tells truth, B must be S. A is the V.
Propositions

- Definition: A *proposition* is a declarative sentence (statement) that is either true (T) or false (F), but not both
  - Fact-based declaration
    - $1 + 1 = 2$
    - “A is not the knave”
    - “If A is a knight, then B is not a knight”
  - Excludes commands, questions and opinions
    - “What time is it?”
    - “Be quiet!”
  - What about statements with (non-constant) variables?
    - $x + 2 = 5$
    - “n is an even number”
Predicates

Definition: A *predicate* is a proposition whose truth depends on one or more variables

- Variables can have various domains:
  - nonnegative integers
  - \(x > 1\)
  - people: “all people on the island are knights, knaves or spies”

- Notation: \(P(x)\)
  - Not an ordinary function!
  - \(P(x)\) is either True or False
Puzzle (propositions)

- Statements:
  - $A$: “I am the knight”
  - $B$: “A is not the knave”
  - $C$: “B is not the knave”

- Let's introduce propositional (boolean) variables:
  - $V_A := “A is the knave”, V_B := “B is the knave”
  - $V_A$ or $V_B := “A is the knave or B is the knave”
  - If $V_A$ then not $V_B := “If A is the knave then B is not the knave”
  - $K(p) := “person p is a knight”$
Constructing Propositions

- English: modify, combine, and relate statements with “not”, “and”, “or”, “implies”, “if-then”

- Atomic propositions: boolean constant (T,F) or variable (e.g. p, q, r, V_A, V_B)

- Compound propositions: apply operators to atomic forms in order of precedence.
  - Construct from logical connectives and other propositions.

- Precise mathematical meaning of operators can be specified by truth tables
Common Operators

- **Negation**: “not” ¬
- **Conjunction**: “and” ∧
- **Disjunction**: “or” ∨
- **Implication/Conditional**: “if-then” →

- **Monadic** operator: one argument
  - Examples: identity, negation, constant ... (4 operators)

- **Dyadic** operator: two arguments
  - Examples: conjunction, disjunction ... (16 operators)
Truth Tables (idea)

- Boolean values & domain: \{T,F\}
- \textit{n-tuple}: \((x_1, x_2, \ldots, x_n)\)
- Operator on n-tuples: \(g(x_1 = v_1, x_2 = v_2, \ldots, x_n = v_n)\)
- Definition: A \textit{truth table} defines an operator ‘\(g\)’ on n-tuples by specifying a boolean value for each tuple
- Number of rows in a truth table?
  - \(R = 2^n\)
- Number of operators with \(n\) arguments?
  - \(2^R\)
Truth Table (negation)

- The *negation* of a proposition \( p \) is denoted by \( \neg p \) and has this truth table:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \neg p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

- **Example:** If \( p \) denotes “The earth is round.”, then \( \neg p \) denotes “It is not the case that the earth is round,” or more simply “The earth is not round.”
Truth Table (conjunction)

- The *conjunction* of propositions $p$ and $q$ is denoted by $p \land q$ and has this truth table:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

- **Example**: If $p$ denotes “I am at home.” and $q$ denotes “It is raining.” then $p \land q$ denotes “I am at home and it is raining.”
Truth Table (disjunction)

- The *disjunction* of propositions $p$ and $q$ is denoted by $p \lor q$ and has this truth table:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
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</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

- **Example:** If $p$ denotes “I am at home.” and $q$ denotes “It is raining.” then $p \lor q$ denotes “I am at home or it is raining.”
Truth Table (exclusive or)

- If only one of the propositions $p$ and $q$ is true but NOT both, we use “Xor” symbol

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \oplus q$</th>
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<tbody>
<tr>
<td>T</td>
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</tr>
</tbody>
</table>

- **Example**: When reading the sentence “Soup or salad comes with this entrée,” we do not expect to be able to get both soup and salad
Truth Table (implication)

- If $p$ and $q$ are propositions, then $p \rightarrow q$ is a conditional statement or implication: “if $p$, then $q$”

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
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</thead>
<tbody>
<tr>
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<td>F</td>
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<td>T</td>
</tr>
</tbody>
</table>

- **Example**: If $p$ denotes “I am at home.” and $q$ denotes “It is raining.” then $p \rightarrow q$ denotes “If I am at home then it is raining.”

- In $p \rightarrow q$, $p$ is the antecedent and $q$ is the consequent
Understanding Implication

- There does not need to be any connection between the antecedent or the consequent.
  - The “meaning” of $p \rightarrow q$ depends only on the truth values of $p$ and $q$.
  - “If pigs fly then you are rich.”

- Think of an obligation or a contract
  - “If I am elected, then I will lower taxes.”
Puzzle (compound propositions)

- **Statements:**
  - $A$: “I am the knight”
  - $B$: “A is not the knave”
  - $C$: “B is not the knave”

- **Compound propositions:**
  - $\neg V_A ::= “A is not the knave”$
  - $K_A \lor K_B ::= “A is the knight or B is the knight”$
  - $V_A \rightarrow \neg V_B ::= “If A is the knave, then B is not the knave”$
  - $K_c \rightarrow \neg V_B ::= “If C is the knight, then C tells the truth”$
Truth Table (rules)

- Row for every combination of values for atomic propositions
- Column for truth value of each expression in the compound proposition
- Column (far right) for the truth value of the compound proposition
- Build step by step
  - $p \lor q \rightarrow \neg r$ means $(p \lor q) \rightarrow \neg r$
- Big problem with this approach!

<table>
<thead>
<tr>
<th>Operator</th>
<th>Precedence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neg$</td>
<td>1</td>
</tr>
<tr>
<td>$\land \lor$</td>
<td>2, 3</td>
</tr>
<tr>
<td>$\rightarrow \leftrightarrow$</td>
<td>4, 5</td>
</tr>
</tbody>
</table>
Truth Table (example)

Construct a truth table for $p \lor q \rightarrow \neg r$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$\neg r$</th>
<th>$p \lor q$</th>
<th>$p \lor q \rightarrow \neg r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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Logical Equivalences

- Two compound propositions $p$ and $q$ are *logically equivalent* if and only if the columns in the truth table giving their truth values agree.
  - We write this as $p \iff q$ or as $p \equiv q$
  - Not an operator! (relation on propositions)

- This truth table shows $\neg p \lor q$ is equivalent to $p \rightarrow q$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$\neg p \lor q$</th>
<th>$p \rightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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Converse, Contrapositive, & Inverse

- Given $p \rightarrow q$,
- The *converse* is: $q \rightarrow p$
- The *contrapositive* is: $\neg q \rightarrow \neg p$
- The *inverse* is: $\neg p \rightarrow \neg q$
- Example: “Raining is a sufficient condition for my not going to town.”
  - Converse: If I do not go to town, then it is raining.
  - Inverse: If it is not raining, then I will go to town.
  - Contrapositive: If I go to town, then it is not raining.
Truth Table (biconditional)

- If $p$ and $q$ are propositions, then $p \iff q$ is a biconditional (IFF) statement: “$p$ if and only if $q$”

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \iff q$</th>
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<tbody>
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</tr>
</tbody>
</table>

- **Example**: If $p$ denotes “I am at home.” and $q$ denotes “It is raining.” then $p \iff q$ denotes “I am at home if and only if it is raining.”
Terminology ($p \rightarrow q$)

- **Simple English:**
  - if $p$, then $q$  \hspace{1cm} \text{$p$ implies $q$}
  - if $p$, $q$  \hspace{1cm} \text{$p$ only if $q$}
  - $q$ unless $\neg p$  \hspace{1cm} \text{$q$ when $p$}
  - $q$ if $p$
  - $q$ whenever $p$  \hspace{1cm} \text{$p$ is sufficient for $q$}
  - $q$ follows from $p$  \hspace{1cm} \text{$q$ is necessary for $p$}

- A **necessary** condition for $p$ is $q$
- A **sufficient** condition for $q$ is $p$

- **Biconditional:**
  - $p$ is **necessary and sufficient** for $q$
  - $p \iff q$
Tautology & Contradiction

- **Tautology** is a proposition which is always true
  - Example: $p \lor \neg p$

- **Contradiction** is a proposition which is always false
  - Example: $p \land \neg p$

- **Contingency** is a proposition which is neither a tautology or a contradiction

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
<th>$p \lor \neg p$</th>
<th>$p \land \neg p$</th>
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<tbody>
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</table>
Laws of Logic

- Trivial laws: identity, double negation
- Express $\land$ and $\lor$ in terms of each other via $\neg$
  \[
  \neg(p \land q) \equiv \neg p \lor \neg q
  \]
- Order & Parenthesis (3,4):
  \[
  \neg(p \lor q) \equiv \neg p \land \neg q
  \]
- Distribute operator (5):
  \[
  p \land q \equiv q \land p
  \]
  \[
  (p \land q) \land r \equiv p \land (q \land r)
  \]
  \[
  (p \lor (q \land r)) \equiv (p \lor q) \land (p \lor r)
  \]
- Laws involving (bi)conditional operators
The Axiomatic Method

- Begin with some assumptions (axioms)
  - Given as true or used to specify the system
- Provide an argument (proof)
  - Sequence (chain) of logical deductions and previous “results” (premises)
  - Ends with the proposition in question (conclusion)
- Important true propositions are called theorems
- Hierarchy of derived truths:
  - Proposition: minor result (theorem)
  - Lemma: preliminary proposition useful for proving later propositions
  - Corollary: a proposition that follows in just a few logical steps from a theorem
Logical Argument

- To provide a logical argument (*proof*):
  - Sequence of *logical deductions* (rules of inference) and previous compound propositions (*premises*)
  - Ends with the proposition in question (*conclusion*)

- A *valid* argument can never leads to incorrect (false) conclusion from correct statements (*premises*)

- *Fallacy*: from true statements to incorrect conclusion

- If some premises untrue: conclusion of valid argument might be false

- Conclusion of fallacy might be true

- If premises are correct & argument is valid, conclusion is correct
Rules of Inference (modus ponens)

- Example:
  - Let $p$ be “It is snowing.”
  - Let $q$ be “I will study discrete math.”
  - “If it is snowing, then I will study discrete math.”
  - “It is snowing.”
  - “Therefore, I will study discrete math.”

- Method of rule validation: record (in a truth table) where all premises are true. If the conclusion is also true in every case, then the rule is valid.
Rules of Inference (fallacy)

- Affirm the consequent, conclude the antecedent

Example:
- Let $p$ be “It is snowing.”
- Let $q$ be “I will study discrete math.”
- “If it is snowing, then I will study discrete math.”
- “I will study discrete math.”
- “Therefore, it is snowing.”
Rules of Inference (modus tollens)

Example:
- Let $p$ be “It is snowing.”
- Let $q$ be “I will study discrete math.”

- “If it is snowing, then I will study discrete math.”
- “I will not study discrete math.”
- “Therefore, it is not snowing.”

Fallacy: deny the antecedent ($p$), conclude the consequent ($q$) is false
Common Rules

- **Addition:**
  \[
  \frac{p}{p \lor q}
  \]
  \[
  \therefore p \lor q
  \]

- **Simplification:**
  \[
  \frac{p \land q}{p \land q}
  \]
  \[
  \therefore q
  \]
  \[
  p \lor q
  \]
  \[
  \frac{\neg p}{\neg p}
  \]
  \[
  \therefore q
  \]

- **Disjunctive-syllogism:**

- **Hypothetical-syllogism:**
  \[
  p \rightarrow q
  \]
  \[
  q \rightarrow r
  \]
  \[
  \therefore p \rightarrow r
  \]
Puzzle (logical argument)

- Statements:
  - A: “I am the knight”  \(K_A\)
  - B: “A is not the knave”  \(\neg V_A\)
  - C: “B is not the knave”  \(\neg V_B\)

- Argument:
  - Suppose A is the Knight (K). Then B tells truth, B must be a spy (S). But C tells truth, can’t be a knave (V)

  - \(K_A \rightarrow \neg V_A \quad ::= \quad \text{“If A is the knight, then A is not the knave”}\)
  - \(\neg V_A \rightarrow (K_B \lor S_B) \quad ::= \quad \text{“If A is not knave, then B is knight or spy”}\)
  - \(\neg V_B \rightarrow (K_C \lor S_C) \quad ::= \quad \text{“If B is not knave, then C is knight or spy”}\)
  - \(S_B \rightarrow \neg (S_A \lor S_C) \quad ::= \quad \text{“If B is the spy then A and C are not spies”}\)
Quantifiers

- **Purpose:** express words such as “all”, “some”
- **Universal Quantifier:** “For all”, \( \forall \)
- **Existential Quantifier:** “There exists”, \( \exists \)
- **Definition:**
  - \( \forall x \ P(x) \) asserts \( P(x) \) is true for **every** \( x \) in the domain
  - \( \exists x \ P(x) \) asserts \( P(x) \) is true for **some** \( x \) in the domain
Quantifiers (examples)

- \( \forall x \, P(x) \): “For all \( x \), \( P(x) \)” or “For every \( x \), \( P(x) \)”
- \( \exists x \, P(x) \): “For some \( x \), \( P(x) \)” or “There is an \( x \) such that \( P(x) \)” or “For at least one \( x \), \( P(x) \)”

Example:
1) \( P(x) \) denotes “\( x > 0 \)”
2) \( Q(x) \) denotes “\( x \) is even”

- For positive integers domain, ‘\( \forall x \, P(x) \)’ is true ‘\( \exists x \, P(x) \)’ is true
- For integers domain, ‘\( \forall x \, P(x) \)’ is false but ‘\( \exists x \, P(x) \)’ is true
- For integers domain, ‘\( \forall x \, Q(x) \)’ is false but ‘\( \exists x \, P(x) \)’ is true
Quantifiers (scope)

- **Rules:**
  - The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
  - Note location of parenthesis:
    - $\forall x \, P(x) \lor Q(x)$ means $(\forall x \, P(x)) \lor Q(x)$
    - $\forall x \, (P(x) \lor Q(x))$ means something different
  - Variable not within scope (clause to which it applies) of any quantifier: *unbound variable*
    - $x + 4 > 2$
    - $\forall y \, [2x + 3y = 7]$
Quantifiers (translation)

- Example:
  1) $P(x)$: “$x$ has taken calculus.”
  2) Domain: students in class
  3) $\forall x 
     P(x)$: “Every student in class has taken calculus.”

Translate: “It is not the case that every student in class has taken calculus.”

Answer:
  1) $\neg \forall x 
     P(x)$ $\neg( \forall x )[ P(x) ]$
  2) “There is a student in class who has not taken calculus”
     $\exists x \neg P(x)$
Quantifiers (negation rules)

- Rules:
  
  \[ \neg \forall x P(x) \equiv \exists x \neg P(x) \]
  
  \[ \neg \exists x P(x) \equiv \forall x \neg P(x) \]
Quantifiers (translation)

- Example:
  1. “All lions are fierce.”
  2. “Some lions do not drink coffee.”
  3. “Some fierce creatures do not drink coffee.”

Translate to predicates:

a. P(x): “x is a lion”
b. Q(x): “x is fierce”
c. R(x): “x drinks coffee”

1. ∀x [ P(x) → Q(x) ]
2. ∃x [ P(x) ∧ ¬R(x) ]
3. ∃x [ Q(x) ∧ ¬R(x) ]
Quantifiers (mixing)

- Nested quantifiers:
  - “Every real number has an inverse”
  - $\forall x \exists y (x + y = 0)$
  - Specify domain when not evident: the domains of $x$ and $y$ are the real numbers
    \[(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[x + y = 0]\]
- Does order matter?
  - Switching order is not safe when the quantifiers are different!
  - $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$ have the same truth value
Nested Quantifiers (translation)

- Example 1: “Brothers are siblings.”
  - **Solution**: $\forall x \forall y [ B(x,y) \rightarrow S(x,y) ]$

- Example 2: “Everybody loves somebody.”
  - **Solution**: $\forall x \exists y L(x,y)$

- Example 3: “There is someone who is loved by everyone.”
  - **Solution**: $\exists y \forall x L(x,y)$
Nested Quantifiers (negation)

- Example 1: “There does not exist a woman who has taken a flight on every airline in the world.”

  ➢ **Solution:** \( \neg \exists w \forall a \exists f \left[ P(w,f) \land Q(f,a) \right] \)

- Use negation rules to move \( \neg \) as far inwards as possible:
Example 1: “There does not exist a woman who has taken a flight on every airline in the world.”

\[ \neg \exists w \forall a \exists f \ [ P(w,f) \land Q(f,a) ] \]

Use negation rules to move \( \neg \) as far inwards as possible:

\[ \forall w \exists a \forall f \ [ \neg P(w,f) \lor \neg Q(f,a) ] \]
Proof Patterns

- **Proof approach:**
  - Direct / Indirect methods
  - Forward / Backward reasoning

- **Standard templates:**
  - Implication (If P then Q)
    - Contrapositive (if not Q then not P)
  - If and only if statement (P if and only if Q)
  - By cases
  - By contradiction
“Backward” Reasoning

- Claim: “arithmetic/geometric means inequality”
- Approach:
  1. Start from conclusion
  2. Show when conclusion is true
  3. Algebraic manipulation
     a. Simplify
  4. Derive simple equivalent premise which is clearly true

Let $a, b > 0$, $a \neq b$.

Then, $\frac{(a + b)}{2} > \sqrt{ab}$

$(a + b) > 2\sqrt{ab}$

$(a + b)^2 > 4ab$

$a^2 + 2ab + b^2 > 4ab$

$a^2 - 2ab + b^2 > 0$

$(a - b)^2 > 0$
Proving the Contrapositive

- **Claim:** “If $r$ is an irrational number then $\sqrt{r}$ is an irrational number”

- **Approach:**
  1. Assume $\sqrt{r}$ is rational, show that $r$ is rational.
  2. Use definition to express $\sqrt{r}$ as a fraction
  3. Algebraic manipulation: square both sides
  4. Conclude claim

$$Q = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$$
Proving If and Only If

Claim: “The standard deviation (std) of a set of numbers is zero if and only if (iff) all the values are equal to the mean”

Approach:

1. Construct chain of iff statements
2. Use definition of std and mean
3. Algebraic manipulation
   a. Simplify: square both sides
4. Show that condition holds for each value iff condition holds for the set
Proof by Cases

- **Claim:** “Let \( x \) be any integer, then \( x^2 + x \) is even”

- **Approach:**
  1. Break into cases:
     a. Case 1: \( x \) is even
     b. Case 2: \( x \) is odd
  2. Use definition of even/odd integer to express \( x^2 + x \) as an even integer
     a. Case 1: \( x = 2n \)
     b. Case 2: \( x = 2n + 1 \)
Proof by Contradiction

Claim: “√2 is irrational”

Approach:
1. Assume √2 is rational
2. Use definition to express √2 as fraction in lowest terms
3. Algebraic manipulation
   a. Square both sides
   b. Apply rules of divisibility
4. Derive a negation of one of the premises (2), that is the contradiction.