Officehours: Check website this afternoon. on zoom this week.

Insertion Sort: $T(n) = c_1 + c_2 n + c_3 n^2$

$\{1, 2, 3 \ldots\}$

Asymptotic notation: f, g are functions from N to nonnegative real numbers.

informal

$f = O(g)$ f has growth rate no faster than g $f \leq g$

$f = \Omega(g)$ - - - - - at least as fast as g $f \geq g$

$f = \Theta(g)$ - - - - - - - the same as g $f = g$

Constant, fixed integer

↳ does not depend on the input length n.

$f = O(g)$: There exist two positive constants c and n_0 such that

$$0 \le \boxed{f(n) \le c \cdot g(n)} \quad \text{for all } n \ge n_0 .$$

$f = \Omega(g)$: There - - - - - - - - - - - - - - - - - -

$$g(n) \le c\, f(n) \quad \text{for all } n \ge n_0 .$$

$f = \Theta(g)$: $f = O(g)$ and $f = \Omega(g)$

There exist two positive constants c and n_0 such that

$$\frac{1}{c} \cdot g(n) \le f(n) \le c \cdot g(n) \quad \text{for all } n \ge n_0 .$$

$T(n) = c_1 + c_2 n + c_3 n^2$

c_1 c_2 c_3 are positive constants.

$T(n) = O(n^2)$

Proof 1: $T(n) = O(n^2)$

set $C = C_1 + C_2 + C_3$ $\quad n_0 = 1$

$$T(n) = \boxed{C_1 + C_2 n + C_3 n^2} \leq \underset{C_1+C_2+C_3}{C} \cdot n^2 \quad \text{for all } n \geq n_0 = 1$$

$\Rightarrow T(n) = O(n^2)$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)}$$

$\boxed{\text{Proof 2}}$: $\displaystyle\lim_{n\to+\infty} \frac{T(n)}{n^2} = \lim_{n\to\infty} \frac{C_1}{n^2} + \frac{C_2}{n} + C_3 = C_3$

$T(n) = \Theta(n^2)$

for any $\varepsilon > 0$, there is an n_0 such that

$\varepsilon = \frac{C_3}{2}$ $\quad \cancel{2C_3}$

$$\left| \frac{T(n)}{n^2} - C_3 \right| \leq \varepsilon \quad \text{for all } n \geq n_0$$

$$0.5C_3 \leq \frac{T(n)}{n^2} \leq C_3 + \varepsilon = 1.5C_3 \quad \text{for all } n \geq n_0$$

$$(0.5C_3)\, n^2 \leq T(n) \leq (1.5C_3) \cdot n^2$$

$O(g)$

$f = o(g)$: For any positive constant c, there is a positive constant n_0 such that $f < g$

$f = \omega(g)$ $f > g$.

$\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$

$f(n) \le c \cdot g(n)$ for all $n \ge n_0$

0.0001

$n^{1.9} = o(n^2)$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^{1.9}}{n^2} = 0$$

For any $\varepsilon > 0$, there is an n_0 such that

$$\left| \frac{n^{1.9}}{n^2} - 0 \right| \le \varepsilon \quad \text{for all } n \ge n_0$$

$$n^{1.9} \le \varepsilon \cdot n^2 \quad \text{for all } n \ge n_0$$

f $\qquad$ g

$\underbrace{\log n \quad \log^{\boxed{2}} n \quad \cdots}_{\text{polylog functions}}$ $\quad$ $\underbrace{n^{0.1} \quad n^{0.2} \cdots n \quad n^2 \cdots n^{100}}_{\text{polynomials}}$ $\quad$ $n^{\log n} \quad n^{\log^2 n} \cdots$ $\quad$ $\underbrace{2^n \quad 3^n \cdots n! \cdots 2^{n^2}}_{\text{exponential}}$

$f = o(g)$

$$\log^{1000} n = o(n^{0.0001})$$

$$\lim_{n \to \infty} \frac{\log^{1000} n}{n^{0.0001}} = 0$$

Correctness of Insertion Sort : Inductions

Statement about $n = 0, 1, 2 \ldots$

$n = 1, 2 \ldots$

Want to show the statement for all $n = 1, 2, 3 \ldots$

(trivial)

Induction : Base case : prove it holds for $n = 1$

Induction Step : Assuming the statement holds for some n, prove it holds for the next case $n+1$.

inductive hypothesis

Conclude that statement holds for all n

$1+2 \cdots + n = \frac{n(n+1)}{2}$

$1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 1$ $(*)$

Base case: Check when $n=1$ LHS $=1$ RHS $=1$ ✓

Induction Step: Assuming $(*)$ holds for n

prove $(*)$ ... $n+1$

$$1^2+2^2 \cdots + n^2 + (n+1)^2 \underset{\text{IH}}{=} \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \overset{?}{=} \frac{(n+1)(n+2)(2n+3)}{6}$$

Lemma: After the ith loop, B contains $a_1 \cdots a_i$ and is sorted in nondecreasing order. (*)

$\Downarrow$

Theorem: Insertion Sort is correct on all inputs

Proof by induction: Base Case: $i=1$ trivial.

Induction Step: Assuming (*) holds after i rounds

prove (*) - - - - $i+1$ rounds.

Concluded (*) holds for all i.

Divide & conquer : Merge Sort (A) $A = \langle a_1 \cdots a_n \rangle$ array of n integers

If $n=1$, return A.

Divide

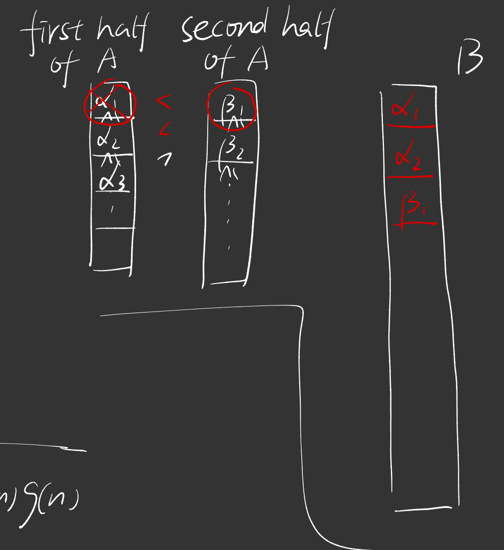
MergeSort$(A[1, \ldots, \lceil \frac{n}{2} \rceil])$

MergeSort$(A[\lceil \frac{n}{2} \rceil + 1, \ldots n])$

conquer

Merge the first half and second half of A into a sorted array B.

linear in the length of A.



$$\underbrace{T(n) = \overbrace{T(\lceil \tfrac{n}{2} \rceil) + T(n - \lceil \tfrac{n}{2} \rceil)}^{2 \cdot T(\frac{n}{2})} + \boxed{\Theta(n)}}_{\text{recurrence}}$$

$$T(1) = \Theta(1)$$

$$T(n) = \Theta(n \lg n)$$

Master theorem

$$h(n) = \sqrt{f(n) g(n)}$$