W3203
Discrete Mathematics

Recurrence Relations

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Outline

- Recurrence Systems (Relations)
- Recurrence Types
- Solving Homogeneous Linear RRs
- Degree 1,2,3 RRs
- Nonhomogeneous Linear RRs
- Divide & Conquer RRs
- Generating Functions (summary)
- Text: Rosen 8.1-8.4
- Text: Lehman 15, 21

Recurrence Systems

- Definition: a finite set of *initial conditions*, and a formula (*recurrence relation*) that expresses a subscripted variable as function of lower-indexed values is a *recurrence system*.
- Notation:
  - *Initial conditions:* $a_0 = c_0,\ a_1 = c_1,\ \ldots,\ a_d = c_d$
  - *Recurrence relation:* $a_n = f(a_{n-1},\ldots,a_0)$
- We wish to find a *solution* to the system i.e., a sequence of values satisfying the initial conditions, and the recurrence relation.

Solving Systems (naive approach)

- Example:
  - *Initial conditions:* $a_0 = 0$
  - *Recurrence relation:* $a_n = a_{n-1} + 2n - 1$
  - *Solution:* $< a_n > = 0, 1, 4, 9, 16, 25$
- Naive approach:
  1. *Work out a few simple cases:*

     $a_1 = a_0 + 2 \cdot 1 - 1 = 1$

     $a_2 = a_1 + 2 \cdot 2 - 1 = 4$
  2. *Spot pattern, guess solution:* $a_n = n^2$
  3. *Prove by induction*

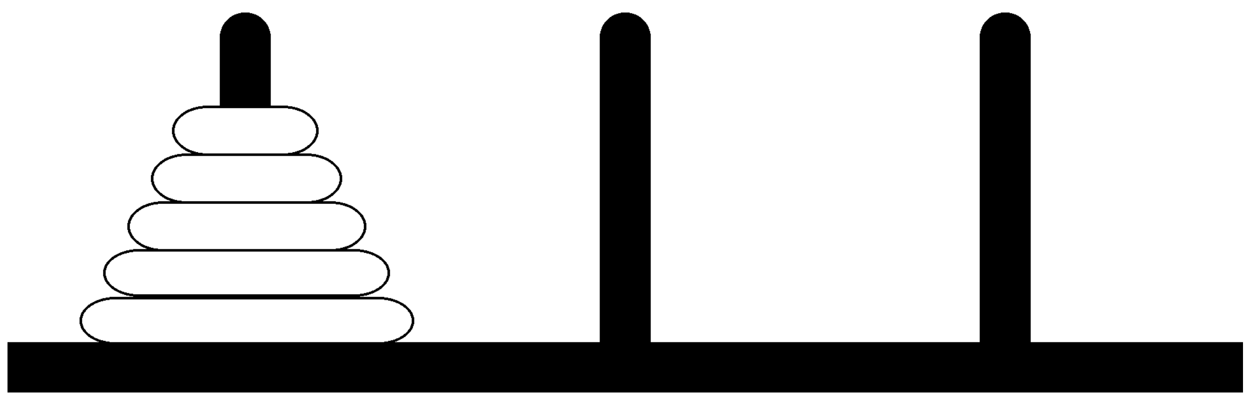
Compound Interest Example

- Deposit \$1 to compound at annual rate ***r***:
  - *Initial conditions:* $a_0 = 1$
  - *Recurrence relation:* $a_n = (1 + r)a_{n-1}$
  - *Solution:* $< a_n > = (1 + r)^n$
- Naive approach:
  1. *Work out a few simple cases:*

     $a_1 = (1 + r)$ $a_2 = (1 + r)^2$ $a_3 = (1 + r)^3$
  2. *Spot pattern, guess solution:* $a_n = (1 + r)^n$
  3. *Prove by induction:* *(base case $n = 0$)*

Tower of Hanoi Example

- What is the minimum number of moves required to move the stack from one rod to another, where smaller disks must stay on top of larger disks?
  - *Initial conditions:* $a_0 = 0$
  - *Recurrence relation:* $a_n = 2a_{n-1} + 1$
  - *Solution:* $< a_n > = 2^n - 1$



Tower of Hanoi (solution)

- Minimum number of moves:
  - *Initial conditions:* $a_0 = 0$
  - *Recurrence relation:* $a_n = 2a_{n-1} + 1$
  - *Solution:* $< a_n > = 2^n - 1$
- Naive approach:
  1. *Work out a few simple cases:*

     $a_1 = 1$ $a_2 = 3$ $a_3 = 7$
  2. *Spot pattern, guess solution:* $a_n = 2^n - 1$
  3. *Prove by induction:* *(base case n = 0)*

Catalan Recursion (example)

- Counting objects:
  - *Initial conditions:* $c_0 = 1$
  - *Recurrence relation:* $c_n = c_0 c_{n-1} + c_1 c_{n-2} + \ldots + c_{n-1} c_0$
  - *Solution:*

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

- Naive approach:
  1. *Work out a few simple cases:*

     $c_1 = c_0 c_{n-1} = 1\cdot 1 = 1$

     $c_2 = c_0 c_1 + c_1 c_0 = 1\cdot 1 + 1\cdot 1 = 2$

     $c_3 = c_0 c_2 + c_1 c_1 + c_2 c_0 = 1\cdot 2 + 1\cdot 1 + 2\cdot 1 = 5$

     $c_4 = \ldots = 1\cdot 5 + 1\cdot 2 + 2\cdot 1 + 5\cdot 1 = 14$
  2. *Spot pattern, guess solution?* *1, 1, 2, 5, 14, 42, ...*
  3. *Proof?*

Recurrence Types

- Some recurrence relations (RRs) have specific types which we can exploit to solve the system
- Definition: a RR has *degree k*, if the lowest term the function depends on is a_{n-k}
- A RR is *linear of degree k* if it has the form (b)
- A RR is *homogeneous* if g(n) = 0
- Notation:
  a. *Degree k:* $a_n = f(a_{n-1},\ldots,a_{n-k})$
  b. *Linear of Degree k:* $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + g(n)$
- Assumptions: ∀j, c_j is a real function, $c_k \neq 0$

Recurrence Types (examples)

- Notation:
  a. *Degree k:* $a_n = f(a_{n-1}, \ldots, a_{n-k})$
  b. *Linear of degree k:* $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + g(n)$

1. Sequence of squares: $a_n = a_{n-1} + 2n - 1$
2. Compound Interest: $a_n = (1 + r)a_{n-1}$
3. Tower of Hanoi: $a_n = 2a_{n-1} + 1$

- *{1,3}: "linear of degree 1 and non-homogeneous"*

4. Catalan numbers: $c_n = c_0 c_{n-1} + c_1 c_{n-2} + \ldots + c_{n-1} c_0$

- *"quadratic, homogeneous and without fixed degree"*

5. Fibonacci numbers: $F_n = F_{n-1} + F_{n-2}$

- *"linear of degree 2 and homogeneous"*

Solving Hom. Linear RRs w. Const. Coeff.

- Given: initial conditions and RR,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

1. Assume general solution form:

$$a_n = r^n$$

2. Substitute into the recurrence:

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$$

3. Cancel powers and rewrite equation:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$$

4. Find roots of characteristic equation:

$$r_1 \;\; r_2 \; \ldots \; r_k$$

5. Write the general solution:

$$a_n = u_1 (r_1)^n + u_2 (r_2)^n + \ldots + u_k (r_k)^n$$

6. Use initial conditions to obtain k linear equations with k unknowns
7. Solve for the unknowns: $u_1 \;\; u_2 \; \ldots \; u_k$

Hom. Linear D1 RR

- Given: initial conditions and RR,

$$a_0 = d$$

$$a_n = c_1 a_{n-1}$$

1. Assume general solution form: $a_n = r^n$
2. Substitute into the recurrence: $r^n = c_1 r^{n-1}$
3. Cancel powers and rewrite equation: $r - c_1 = 0$
4. Find roots of characteristic equation: $r_1 = c_1$
5. Write the general solution: $a_n = u_1(r_1)^n = u_1(c_1)^n$
6. Use initial conditions to obtain 1 equation with 1 unknown:

$$a_0 = u_1(c_1)^0 \quad \rightarrow \quad d = u_1$$

7. Solve for the unknowns: $a_n = d(c_1)^n$

Compound Interest (general example)

- Given: initial conditions and RR,

$$a_0 = C$$

$$a_n = (1+R)a_{n-1}$$

1. Assume general solution form: $a_n = r^n$
2. Substitute into the recurrence: $r^n = c_1 r^{n-1}$
3. Cancel powers and rewrite equation: $r - c_1 = 0$
4. Find roots of characteristic equation: $r_1 = 1+R$
5. Write the general solution: $a_n = u_1(r_1)^n = u_1(1+R)^n$
6. Use initial conditions to obtain 1 equation with 1 unknown:

$$a_0 = u_1(1+R)^0 \quad \rightarrow \quad C = u_1$$

7. Solve for the unknowns: $a_n = C(1+r)^n$

Hom. Linear D2 RR

- Given: initial conditions and RR,

$$a_0 = d_0 \quad a_1 = d_1$$
$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

1. Assume general solution form: $a_n = r^n$
2. Substitute into the recurrence: $r^n = c_1 r^{n-1} + c_2 r^{n-2}$
3. Cancel powers and rewrite equation: $r^2 - c_1 r - c_2 = 0$
4. Find roots of characteristic equation: r_1, r_2
5. Write the general solution: $a_n = u_1(r_1)^n + u_2(r_2)^n$
6. Use initial conditions to obtain 2 eqs with 2 unknowns:

   (1) $a_0 = u_1(r_1)^0 + u_2(r_2)^0 \rightarrow u_1 + u_2 = d_0$

   (2) $a_1 = u_1(r_1)^1 + u_2(r_2)^1 \rightarrow r_1 u_1 + r_2 u_2 = d_1$
7. Solve for the unknowns:

Fibonacci Recurrence

- Given: initial conditions and RR,

  $$a_0 = 0 \qquad a_1 = 1$$

  $$a_n = a_{n-1} + a_{n-2}$$

1. Assume general solution form: $a_n = r^n$
2. Substitute into the recurrence: $r^n = r^{n-1} + r^{n-2}$
3. Cancel powers and rewrite equation: $r^2 - r - 1 = 0$
4. Find roots of characteristic equation: $r_{1,2} = (1 \pm \sqrt{5})/2$
5. Write the general solution: $a_n = u_1(r_1)^n + u_2(r_2)^n$
6. Use initial conditions to obtain 2 eqs with 2 unknowns:

   (1) $a_0 = u_1(r_1)^0 + u_2(r_2)^0 \rightarrow u_1 + u_2 = 0$

   (2) $a_1 = u_1(r_1)^1 + u_2(r_2)^1 \rightarrow r_1u_1 + r_2u_2 = 1$
7. Solve for the unknowns:

Fibonacci Recurrence (solution)

- Given: initial conditions and RR,

$$a_0 = 0 \qquad a_1 = 1 \qquad\qquad a_n = a_{n-1} + a_{n-2}$$

4. Find roots: $r_{1,2} = (1 \pm \sqrt{5})/2$
5. General solution: $a_n = u_1(r_1)^n + u_2(r_2)^n$
6. System with 2 unknowns:

   (1) $u_1 = -u_2$

   (2) $r_1 u_1 + r_2 u_2 = 1 \rightarrow (1+\sqrt{5})u_1 - (1-\sqrt{5})u_1 = 2$
7. Solve for the unknowns: $u_1 = \frac{1}{\sqrt{5}}, \quad u_2 = -\frac{1}{\sqrt{5}}$

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{1}{2^n\sqrt{5}}\left[(1+\sqrt{5})^n - (1-\sqrt{5})^n\right]$$

Hom. Linear D3 RR

- Given: initial conditions and RR,

$$a_0 = d_0 \quad a_1 = d_1 \quad a_2 = d_2$$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}$$

1. General form: $a_n = r^n$
2. Substitute: $r^n = c_1 r^{n-1} + c_2 r^{n-2} + c_3 r^{n-3}$
3. Cancel powers: $r^3 - c_1 r^2 - c_2 r - c_3 = 0$
4. Find roots: r_1, r_2, r_3
5. General solution: $a_n = u_1 (r_1)^n + u_2 (r_2)^n + u_3 (r_3)^n$
6. Solve system with 3 unknowns:

(1) $u_1 + u_2 + u_3 = d_0$

(2) $r_1 u_1 + r_2 u_2 + r_3 u_3 = d_1$

(2) $(r_1)^2 u_1 + (r_2)^2 u_2 + (r_3)^2 u_3 = d_2$

Solving **Non**-Hom. Linear RRs

- Given: initial conditions and RR,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + g(n)$$

1. Set $g(n) = 0$, obtain "homogeneous solution" from the associated hom. linear relation:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \quad \rightarrow \quad a_n^{(H)} = u_1(r_1)^n + u_2(r_2)^n + \ldots + u_k(r_k)^n$$

Solving **Non**-Hom. Linear RRs

- Given: initial conditions and RR,

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + g(n)$$

1. Set $g(n) = 0$, obtain "hom. solution" from the hom. linear RR:

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} \rightarrow \quad a_n^{(H)} = u_1(r_1)^n + u_2(r_2)^n + \ldots + u_k(r_k)^n$$

2. Ignore initial conditions and find "particular solution" with the same form as term g(n): $\quad a_n^{(P)} = f(n)$
   - If g(n) is a polynomial, "guess" polynomials of ≥ degree
   - If g(n) is exponential, "guess" exponentials (multiplied by n)
   - Substitute into recurrence and find coefficients

Solving **Non**-Hom. Linear RRs

- Given: initial conditions and RR,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + g(n)$$

1. Set $g(n) = 0$, obtain "hom. solution" from the hom. linear RR:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \rightarrow \quad a_n^{(H)} = u_1(r_1)^n + u_2(r_2)^n + \dots + u_k(r_k)^n$$

2. Ignore initial conditions and find "particular solution" with the same form as term g(n): $a_n^{(P)} = f(n)$
   - If g(n) is a polynomial, "guess" polynomials of equal or higher degree
   - If g(n) is exponential, "guess" exponentials (multiplied by n)
   - Substitute into recurrence and find coefficients

4. Add particular + hom. solutions together to get general solution:

$$a_n^{(G)} = a_n^{(H)} + a_n^{(P)}$$

5. Use initial conditions to obtain k linear equation with k unknowns
6. Solve for the unknowns: $u_1 \; u_2 \; \dots \; u_k$

Tower of Hanoi (general approach)

- Given: initial conditions and RR,

$$a_0 = 0 \qquad a_n = 2a_{n-1} + 1$$

1. Set $g(n) = 0$, obtain "hom. solution" from the hom. linear RR:

$$a_n = 2a_{n-1} \rightarrow a_n^{(H)} = u_1(r_1)^n = u_1 2^n$$

2. Find "particular solution" with the same form as term g(n):

$$a_n^{(P)} = f(n) = K \rightarrow K = 2K + 1 \rightarrow K = -1$$

4. Add particular + hom. solutions together to get general solution:

$$a_n^{(G)} = a_n^{(H)} + a_n^{(P)} = u_1 2^n - 1$$

5. Use initial conditions to obtain 1 linear eq with 1 unknown:

(1) $a_0 = u_1(2)^0 - 1 \rightarrow u_1 = 1$

Non-Hom. Linear RRs (example)

- Given: initial conditions and RR,

$$a_1 = 3 \qquad a_n = 3a_{n-1} + 2n$$

1. Set $g(n) = 0$, obtain "hom. solution" from the hom. linear RR:

$$a_n = 3a_{n-1} \rightarrow a_n^{(H)} = u_1(r_1)^n = u_1 3^n$$

2. Find "particular solution" with the same form as term g(n):

$$a_n^{(P)} = f(n) = cn + d \rightarrow cn + d = 3[c(n-1)+d] + 2n$$

$$\rightarrow 0 = (2c+2)n + (2d-3c) \rightarrow c = -1 \quad d = -3/2$$

4. Add particular + hom. solutions together to get general solution:

$$a_n^{(G)} = a_n^{(H)} + a_n^{(P)} = u_1 3^n - n - 3/2$$

5. Use initial conditions to obtain 1 linear eq with 1 unknown:

$$(1)\ a_1 = u_1(3)^1 - 1 - 3/2 \quad \rightarrow u_1 = 11/6$$

Repeated Roots

- Given: initial conditions and RR,

$$a_0 = d_0 \quad a_1 = d_1$$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

1. Assume general solution form: $a_n = r^n$
2. Substitute into the recurrence: $r^n = c_1 r^{n-1} + c_2 r^{n-2}$
3. Cancel powers and rewrite equation: $r^2 - c_1 r - c_2 = 0$
4. Find roots of characteristic equation: r_1, r_2

What if $r_1 = r_2$? We need approach for repeated roots as general solution [$a_n = u_1(r_1)^n + u_2(r_2)^n$] should be different

Repeated Roots Approach

- Given: initial conditions and RR,

$$a_0 = d_0 \quad a_1 = d_1$$
$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

1. Assume general solution form: $a_n = r^n$
2. Substitute into the recurrence: $r^n = c_1 r^{n-1} + c_2 r^{n-2}$
3. Cancel powers and rewrite equation: $r^2 - c_1 r - c_2 = 0$
4. Find roots of characteristic equation: R *(repeated)*
5. Write the general solution: $a_n = u_1(R)^n + u_2 n(R)^n$
6. Use initial conditions to obtain 2 eqs with 2 unknowns:

   (1) $a_0 = u_1(R)^0 + 0 \cdot u_2(R)^0 \rightarrow u_1 = d_0$

   (2) $a_1 = u_1(R)^1 + 1 \cdot u_2(R)^1 \rightarrow Ru_1 + Ru_2 = d_1$
7. Solve for the unknowns:

Divide-and-Conquer RRs

- Definition: a *divide-and-conquer recurrence* has the following form (where b is a positive integer):
  - *Base case:* $a_0 = c_0$
  - *Recurrence relation:* $a_n = f(n) = C f(n / b) + g(n)$
- Examples:
  - Binary search: C = 1, b = 2, g(n) = c
  - Mergesort: C = 2, b = 2, g(n) = n

Generating Functions

- Definition: an *(ordinary) generating function* for a sequence $a_0, a_1, \ldots$ is a function whose formal power series has a matching sequence of coefficients
- Example 1:
  - < a_n > = 1,1,1,1,...
  - *Generating function (GF):* $$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{j=0}^{\infty} x^j$$
- Example 2:
  - < a_n > = 1,2,4,8,...
  - GF: $$\frac{1}{1-2x} = 1 + 2x + 4x^2 + \ldots = \sum_{j=0}^{\infty} 2^j x^j$$

Constructing Generating Functions

- Given a sequence, how do we construct the generating function?

$$f(x) = \sum_{j=0}^{\infty} a_j x^j \quad g(x) = \sum_{j=0}^{\infty} b_j x^j$$

$$(f+g)(x) = f(x) + g(x) = \sum_{j=0}^{\infty} (a_j + b_j) x^j$$

- Example 1:
  - $< a_n >$ = 0,1,3,7,15,...

$$\frac{1}{1-2x} = \sum_{j=0}^{\infty} 2^j x^j \qquad \frac{-1}{1-x} = \sum_{j=0}^{\infty} (-1) x^j$$

$$GF(x) = \frac{1}{1-2x} - \frac{1}{1-x} = \frac{x}{(1-x)(1-2x)}$$