W3203
Discrete Mathematics

Relations

Spring 2015
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Outline

- Recap
- Representation of relations
- Relation properties: reflexive, symmetric, transitive
- Digraphs
- Equivalence relations
- Partial orders
- Lattice
- Text: Rosen 9

Binary Relations (recap)

- Definition: a *binary relation* R consists of two sets, A (*domain* of R), B, (*codomain* of R), and a subset of A × B called the *graph of* R
- We use "*a R b*", to mean that the pair (a,b) is in the graph of R
- Note: the *Cartesian Product* of two sets (A × B) is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$

N-ary Relations

- Generalization of binary relations to N sets
- Note: the *Cartesian Product* of the sets ($A_1 \times A_2 \times \ldots\ldots \times A_n$) is the set of ordered n-tuples ($a_1, a_2, \ldots\ldots, a_n$) where ∀i, $a_i \in A_i$

$$A_1 \times A_2 \times \cdots \times A_n = \\ \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Representation

Example 9.1.7: The relation Q from the set $\{1,2,3\}$ to the set $\{A,B,C\}$, with the ordered-pairs model

$$Q = \{(1,A),\ (1,B),\ (2,C),\ (3,A),\ (3,C)\}$$

has the ***lists-of-relatives model***

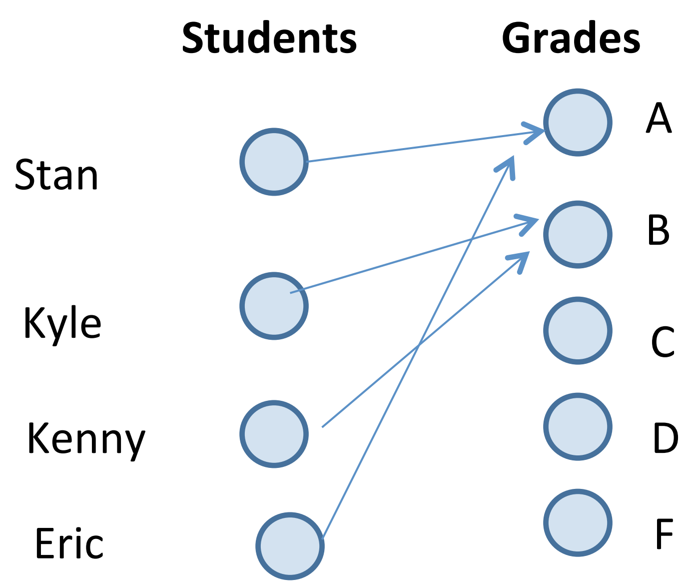
$$1 : A, B$$
$$2 : C$$
$$3 : A, C$$

and the ***matrix model***

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

Representation (digraphs)

- Definition: a *directed graph (digraph)* is a graph or set of *nodes (vertices)* connected by *edges (arcs)*
  - The edges have a direction associated with them
- Digraph representation of binary relations:



Composition of Binary Relations

DEF: Let Q be a relation from S to T and R a relation from T to U. Their ***composition*** $Q \circ R$ is the relation on $S \times U$ that is true for any pair (s, u) such that

$$(\exists t \in T)[Q(s,t) \wedge R(t,u)]$$

Example 9.1.8: Construct $Q \circ R$, where

$$Q = \{(1,A),\ (1,B),\ (2,C),\ (3,A),\ (3,C)\}$$

and

$$R = \{(A,x),\ (A,y),\ (A,z),\ (B,w),\ (B,y)\}$$

Composition (example)

Example 9.1.8: Construct $Q \circ R$, where

$$Q = \{(1, A),\ (1, B),\ (2, C),\ (3, A),\ (3, C)\}$$

and

$$R = \{(A, x),\ (A, y),\ (A, z),\ (B, w),\ (B, y)\}$$

$$\underline{Q} \qquad \underline{R}$$

$$\left\{\begin{array}{l} 1 : A, B \\ 2 : C \\ 3 : A, C \end{array}\right\} \circ \left\{\begin{array}{l} A : x, y, z \\ B : w, y \\ C : \emptyset \end{array}\right\}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} A \quad B \quad C \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \times \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} w \quad x \quad y \quad z \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Composition (solution)

Example 9.1.8: Construct $Q \circ R$, where

$$Q = \{(1,A),\ (1,B),\ (2,C),\ (3,A),\ (3,C)\}$$

and

$$R = \{(A,x),\ (A,y),\ (A,z),\ (B,w),\ (B,y)\}$$

$$\underline{Q} \qquad\qquad \underline{R}$$

$$\left\{\begin{array}{l} 1 : A, B \\ 2 : C \\ 3 : A, C \end{array}\right\} \circ \left\{\begin{array}{l} A : x, y, z \\ B : w, y \\ C : \emptyset \end{array}\right\}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \times \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\underline{Q \circ R}$$

$$= \left\{\begin{array}{l} 1 : w, x, y, z \\ 2 : \emptyset \\ 3 : x, y, z \end{array}\right\}$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Powers of Relations

- Let R be a relation on a set S. Then the powers of R are defined as:

  $R^0 = I$ $\qquad\qquad$ $R^{n+1} = R^n \circ R$

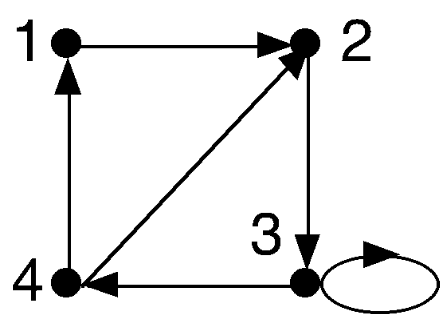
Relation Powers with Digraphs

- The powers of a relation R on a set S:

$$R^0 = I \qquad R^{n+1} = R^n \circ R$$

Example 9.3.2: Consider this relation:

$$R = \{(1,2), (2,3), (3,3), (3,4), (4,1)\}$$



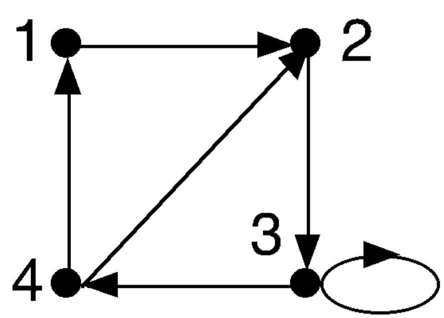
Relation Powers (computing)

- The powers of a relation *R* on a set *S*:

$R^0 = I \qquad R^{n+1} = R^n \circ R$

Example 9.3.2: Consider this relation:

$$R = \{(1,2),(2,3),(3,3),(3,4),(4,1)\}$$



$$R^2 = \{(1,3),(2,3),(2,4),(3,1),(3,3),(3,4),(4,2)\}$$

$$R^3 = \{(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,3),(4,4)\}$$

Properties of Relations

- Let *R* be a relation on a set *S*. The relation is,

$$\textit{Reflexive} \text{ IFF: } \quad (\forall x \in S)\big[R(x,x)\big]$$

$$\textit{Symmetric} \text{ IFF: } (\forall x, y \in S)\big[R(x,y) \to R(y,x)\big]$$

$$\textit{Transitive} \text{ IFF: } \quad (\forall x, y, z \in S)\big[R(x,y) \wedge R(y,z) \to R(x,z)\big]$$

$$\textit{Antisymmetric} \text{ IFF: } (\forall x, y \in S)\big[R(x,y) \wedge R(y,x) \to x = y\big]$$

- Example: Inequality relation on real numbers.
  - Reflexive, *nonsymmetric, and* transitive
  - *Antisymmetric*

Properties (representation)

$$\textit{Reflexive} \text{ IFF: } \quad (\forall x \in S)\big[R(x,x)\big]$$

$$\textit{Symmetric} \text{ IFF: } (\forall x,y \in S)\big[R(x,y) \rightarrow R(y,x)\big]$$

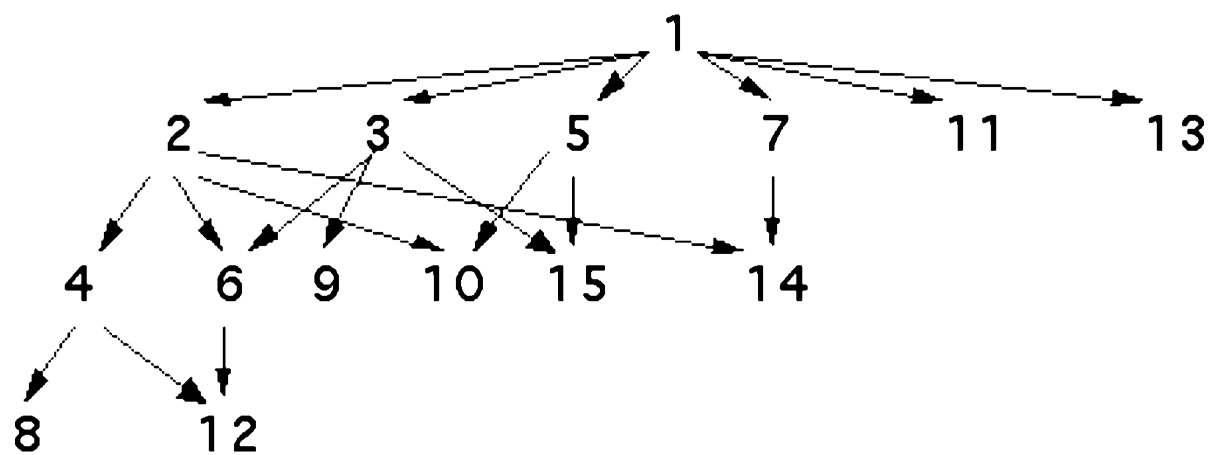
$$\textit{Transitive} \text{ IFF: } \quad (\forall x,y,z \in S)\big[R(x,y) \wedge R(y,z) \rightarrow R(x,z)\big]$$

$$\textit{Antisymmetric} \text{ IFF: } (\forall x,y \in S)\big[R(x,y) \wedge R(y,x) \rightarrow x = y\big]$$

- List of relatives model:
  - *Reflexive*: every member of the domain is listed as one of its own relatives.
  - *Symmetric*: for every pair of elements x,y, each occurs in the list of the other.
- Matrix model:
  - *Reflexive*: 1's down the main diagonal.
  - *Symmetric*: matrix is symmetric around the main diagonal.
  - *Transitive*: nth power of R is a subset of R.

Digraph Representation (example)

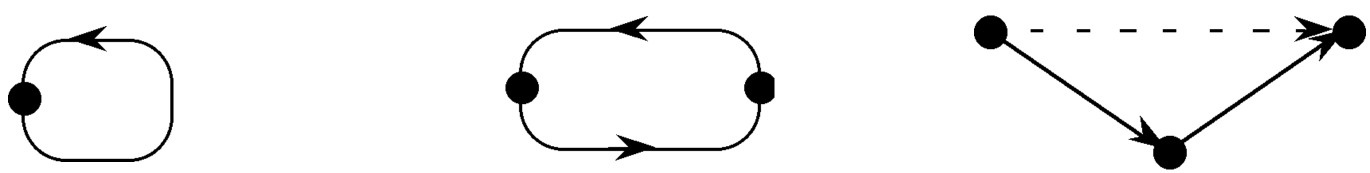
- Let *R* be the relation on a set *S = {1,2,...,14}* defined as:
  1. x properly divides y.
  2. There is no integer u such that x properly divides u and u properly divides y.



- Proper divisibility relation: not reflexive, not symmetric, and not transitive

Digraph Representation (properties)

- Consider the digraph for the relation *R* on a set *S:*
  1. R is reflexive IFF there is a *self loop* at every vertex.
  2. R is symmetric IFF for each arc from-x-to-y there is an arc from-y-to-x.
  3. R is transitive IFF for each *directed path* from-x-to-y there is also an arc directly from-x-to-y.
  4. R is antisymmetric IFF given an arc from-x-to-y ($x \neq y$), there is no arc from-y-to-x.



Equivalence Relations

- Definition: an *equivalence relation* is a binary relation that is reflexive, symmetric, and transitive

Example 9.5.1: Set $S = \{a, b, c, d, e, f\}$ and

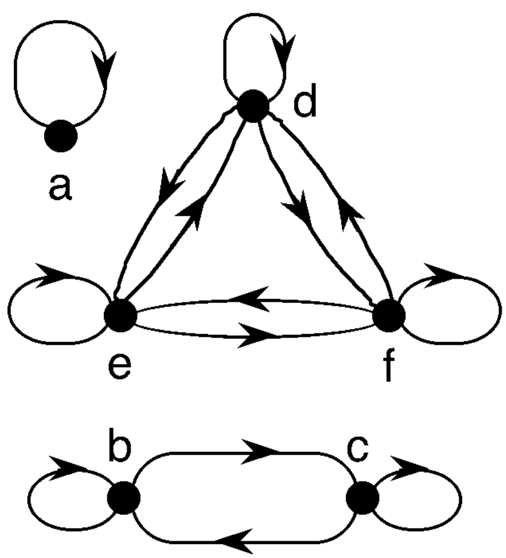
$$R = \{\ (a,a),(b,b),(b,c),(c,b),(c,c),(d,d),(d,e),$$
$$(d,f),(e,d),(e,e),(e,f),(f,d),(f,e),(f,f)\ \}$$

Equivalence Relations (example)

- Binary relation that is reflexive, symmetric, and transitive

Example 9.5.1: Set $S = \{a, b, c, d, e, f\}$ and

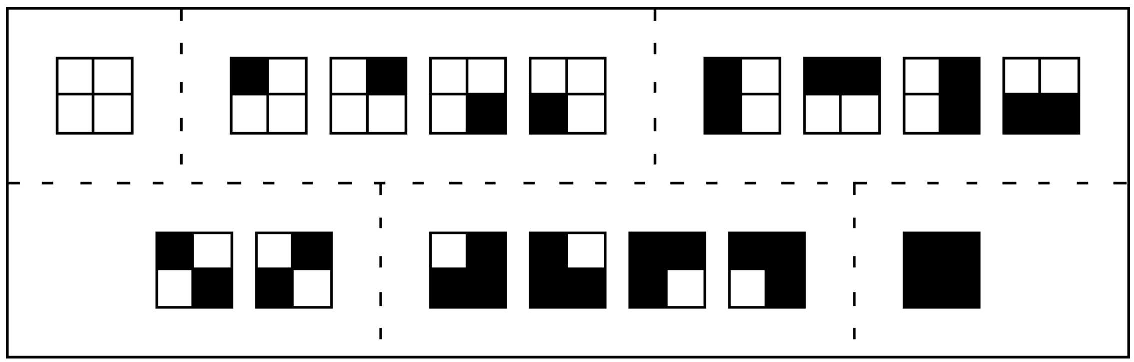
$$R = \{\ (a,a),(b,b),(b,c),(c,b),(c,c),(d,d),(d,e),$$
$$(d,f),(e,d),(e,e),(e,f),(f,d),(f,e),(f,f)\ \}$$



	a	b	c	d	e	f
a	1					
b		1	1			
c		1	1			
d				1	1	1
e				1	1	1
f				1	1	1

Equiv. Classes (geometry example)

- Consider 2-by-2 colored boards. Two boards are related if one can be obtained from the other by rotation or reflection.



Equiv. Classes (fractions & congruence)

Example 9.5.4: domain = rational fractions

$$\left\{ \frac{p}{q} \;\middle|\; p, q \in \mathcal{Z}, q \neq 0 \right\}$$

Then $\dfrac{a}{b}$ and $\dfrac{c}{d}$ are related if $ad = bc$

The partition cells are rational fractions of equal value.

Example 9.5.5: domain $\mathcal{Z}$

eq. rel. = congruence mod 3

Equivalence Classes:

$$[0]_3 = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$$

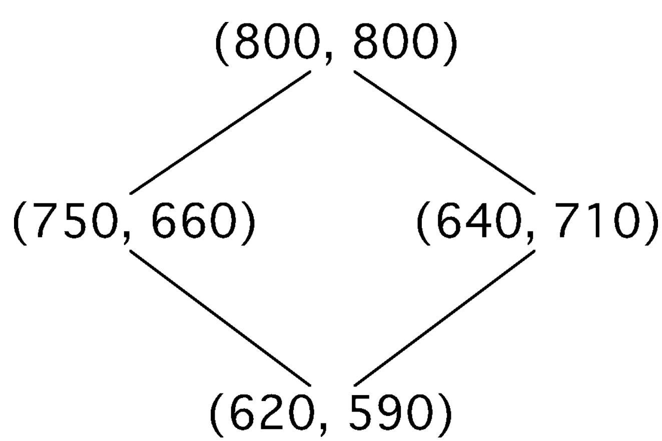
$$[1]_3 = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$$

$$[2]_3 = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$$

Partial Order

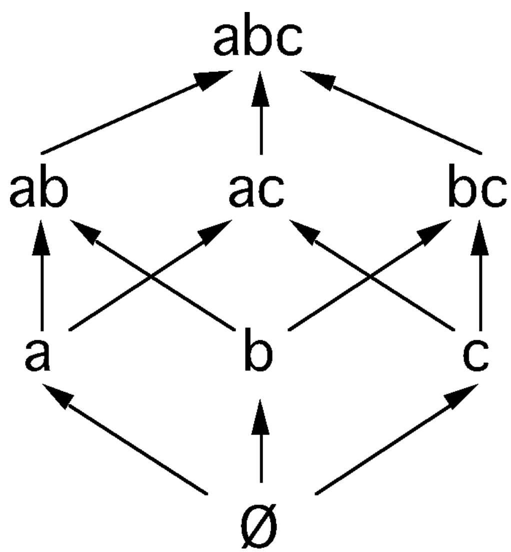
- Definition: a (weak) *partial order* $\preceq$ is a binary relation that is reflexive, antisymmetric, and transitive.
- Definition: a *partial ordered set (poset)* $\langle S, \preceq \rangle$ is a set together with a partial order on it

Example 9.6.1: domain: SAT scores (M, V)
relation: double domination



Poset (subset example)

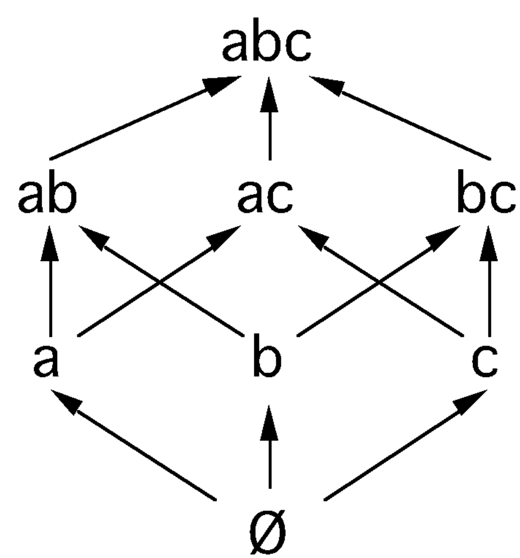
- We write "x $\prec$ y" if "x $\preceq$ y" and x ≠y
- Consider: X = {a,b,c}, then the poset (X, ⊆) is shown below



Transitive	$\dfrac{A \subseteq B \quad B \subseteq C}{A \subseteq C}$
Reflexive	$A \subseteq A$
Antisymmetric	$\dfrac{A \subseteq B \quad B \subseteq A}{A = B}$

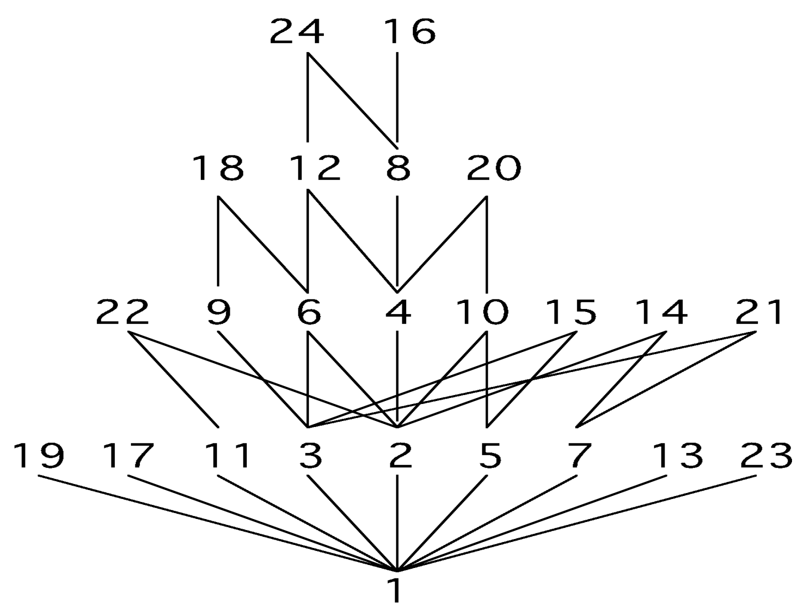
(In)Comparable Elements

- Definition: two elements x, y from a poset (X, R) are *comparable* if either *xRy* or *yRx* and *incomparable* otherwise.
- Subset example: no line connects {a,b} and {b,c} because neither is a subset of the other. We say these two sets are *incomparable* in that ordering.



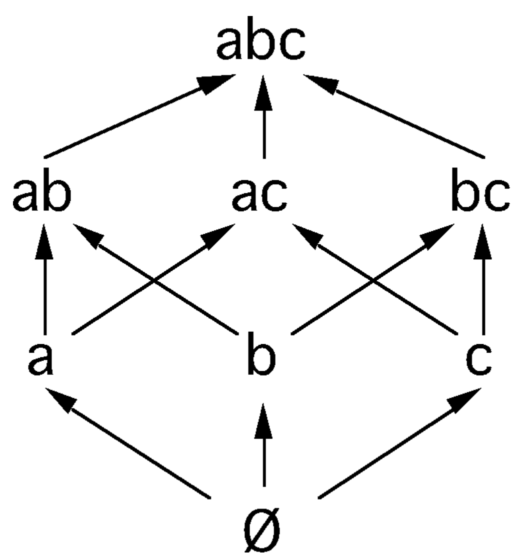
Poset (divisibility example)

- Consider: X = {1 ≤ n ≤ 24}, and the divisibility relation "|".
- An element ***y*** covers an element ***x*** in a poset if "**x** $\prec$ **y**" and there is no element ***u*** such that "**x** $\prec$ **u** $\prec$ **y**"
- The *Hasse Diagram* for (X, |) only shows cover relations



Linear Extension

- Definition: a *linear order* has no *incomparable* elements. Every pair of element is comparable.
- Example: ≤ relation on numbers
- Definition: a *linear extension* of a poset (X, R) is a total ordering Q on S such that $R \subseteq Q$.



linear extensions:

$$\emptyset \leq a \leq b \leq c \leq ab \leq ac \leq bc \leq abc$$

$$\emptyset \leq a \leq b \leq ab \leq c \leq ac \leq bc \leq abc$$