# W3203
# Discrete Mathematics

## Graph Theory

Spring 2015
Instructor: Ilia Vovsha

http://www.cs.columbia.edu/~vovsha/w3203

# Outline

- Graph terminology
- Simple graphs
- Graph types
- Representation of graphs
- Isomorphism
- Connectivity
- Eulerian and Hamiltonian graphs
- Planar graphs
- Coloring
- Text: Rosen 10,
- Text: Lehman 11,12

# Graph Theory

- Substantial amount of research and resources.
- Algorithms and off-the-shelf software available in every programming language.
- Terminology and notation differ between sources.

# Graph Terminology

- Definition: a *graph* G = (V,E) consists of two sets, V – a set of *vertices*, and E – a set of *edges*.
- An edge joins two vertices called its *endpoints*.
- Two vertices are *adjacent* if there is an edge joining them.

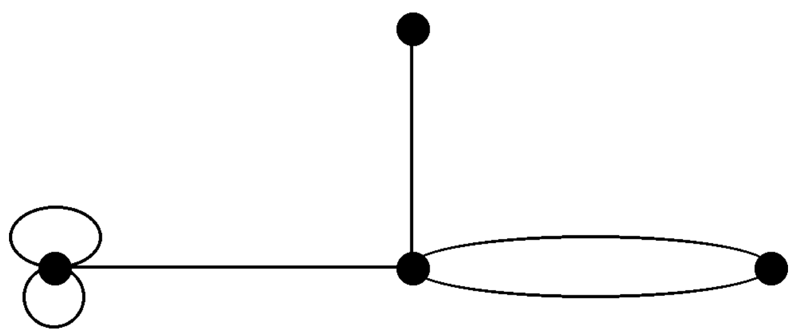


**Fig 10.1.1** A general graph.

# More Graph Terminology

- Definition: a *graph* G = (V,E) is *simple* if:
  1. No self-loops
  2. There is at most one edge between any pair of vertices
- An edge has *direction* if an arrow is added to designate what is "forward"
- An *arc* is an edge with a direction.
- A *digraph (directed graph)* is a collection of vertices and arcs.

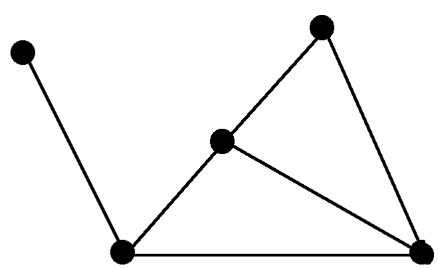


**Fig 10.1.2** A simple graph.

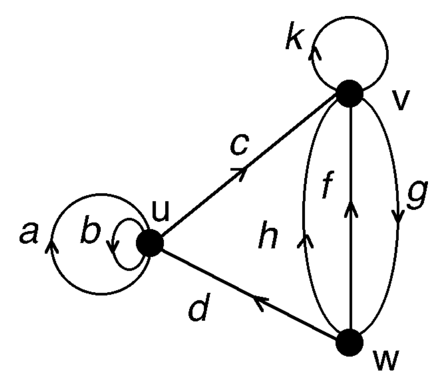
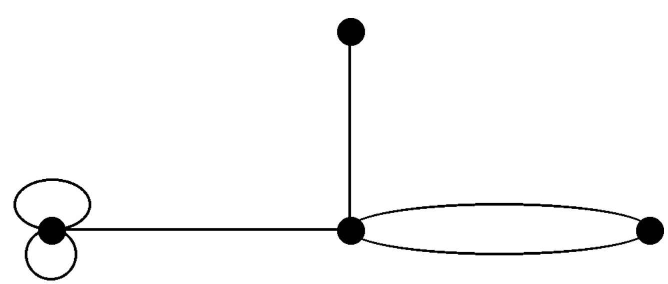



**Fig 10.1.3** A digraph.

# Vertex Degrees

- Definition: the *degree (valence)* of vertex *v* is the number of edge-ends incident on *v* (self-loops contribute 2).
- Definition: the *degree sequence* of graph *G* is a list of all degrees in ascending order.

$$1, \quad 2, \quad 4, \quad 5$$

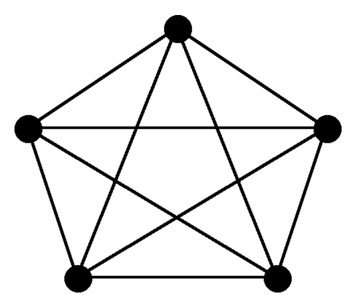


# Vertex Degrees (theorems)

- Theorem (Euler): Σ (degrees of graph) = 2 × (# edges).
- Corollary: a graph has an even number of vertices with odd degree.
- Theorem: If G is a simple graph with at least two vertices, then G has two vertices with the same degree.
- Proof?

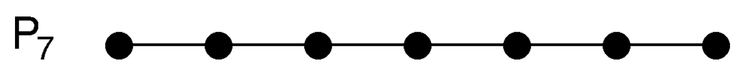
# Complete Graph

- Definition: a *complete graph* is a simple graph with every pair of vertices joined by an edge.
- Notation: $K_n$ denotes the complete graph on n vertices.



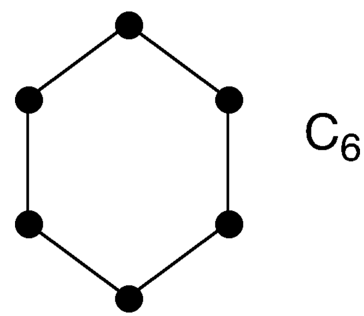
# Path Graph

- Definition: a *path graph* has vertices $\{v_1, v_2, \ldots, v_n\}$ and edges $\{e_1, e_2, \ldots, e_{n-1}\}$ such that edge $e_k$ joins vertices $\{v_k, v_{k+1}\}$.
- Notation: $P_n$ denotes the path graph on n vertices.



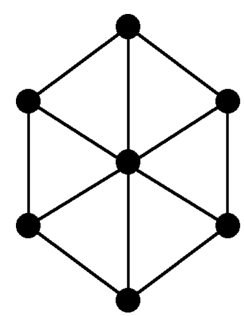
# Cycle Graph

- Definition: a *cycle graph* has vertices $\{v_1, v_2, \ldots, v_n\}$ and edges $\{e_1, e_2, \ldots, e_n\}$ such that edge $e_k$ joins vertices $\{v_k, v_{k+1}\}$ where $v_{k+1}$ is "mod n".
- Notation: $C_n$ denotes the cycle graph on n vertices.



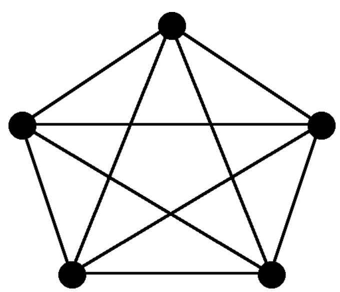
# Wheel Graph

- Definition: a *wheel graph* is a cycle graph with an additional *hub vertex* joined to every other vertex.
- Notation: $W_n$ denotes the wheel graph on n+1 vertices.

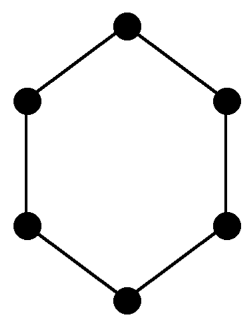


# Regular Graph

- Definition: a graph (simple or not) is *regular* if every vertex has the same degree
- Examples:
  1. $K_n$ is regular of degree n-1.
  2. $C_n$ is regular of degree 2.
  3. $W_3$ is regular of degree 3.



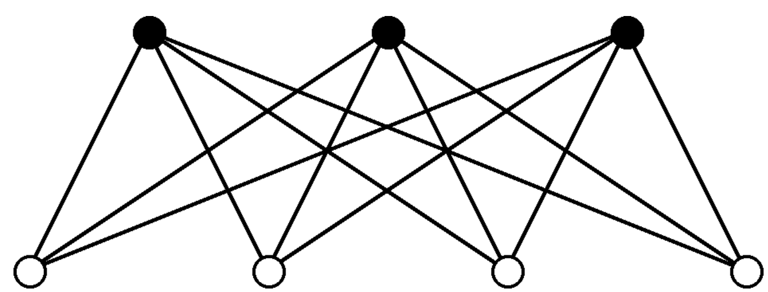
$K_5$



$C_6$

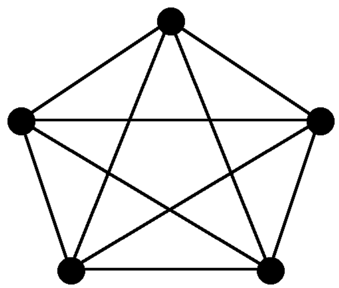
# Bipartite Graph

- Definition: a *bipartite graph* has a vertex set which can be partitioned into two disjoint subsets such that every edge joins a vertex in subset.1 to a vertex in subset.2
- Definition: a simple graph is *complete bipartite graph* if it is bipartite and every vertex in one subset is joined to every vertex in the other.
- Notation: $K_{n,m}$ denotes the complete bipartite graph with n+m vertices.

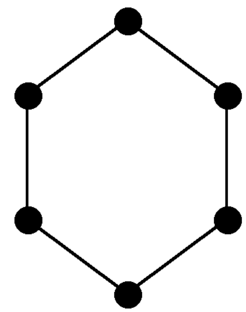


$K_{3,4}$

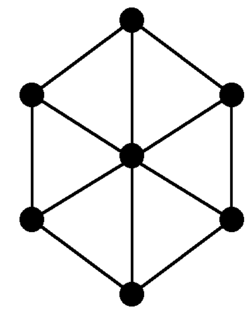
# Graph Types



$K_5$

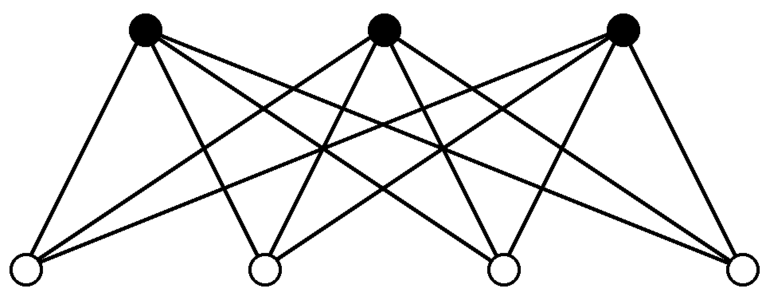


$C_6$



$W_6$

$P_7$



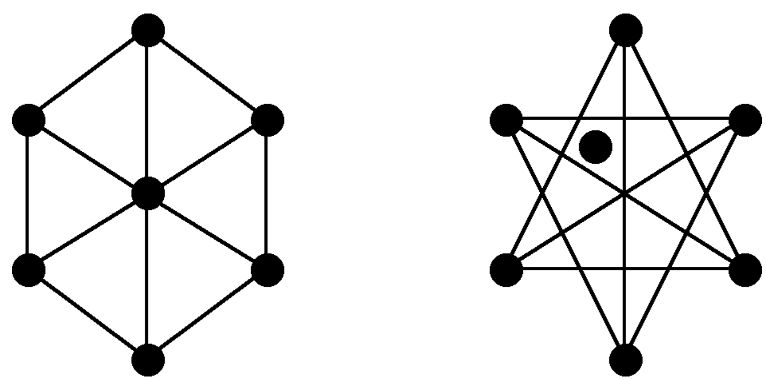
$K_{3,4}$

# Subgraphs

- Definition: a *subgraph* of a graph G = (V,E) is a graph H = (U,D) such that $U \subseteq V$ and $D \subseteq E$.
- Remark: since H is a graph, U must contain all the endpoints of the edges in D.

# Edge-Complement Graph

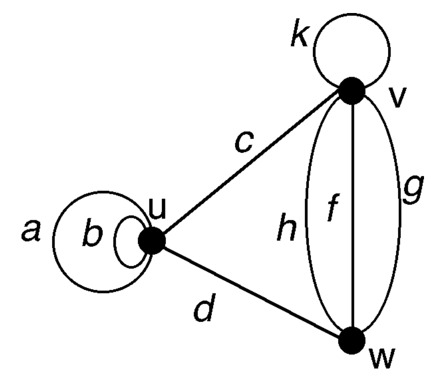
- Definition: the *edge-complement* of a simple graph G is a graph $G^C$ on the same vertex set as G, where two vertices are joined by an edge IFF they are not adjacent in G.



# Graph Representation (incidence table)

- *Incidence table* representation: columns indexed by each edge, endpoints are entries.

**Example 10.3.1:**



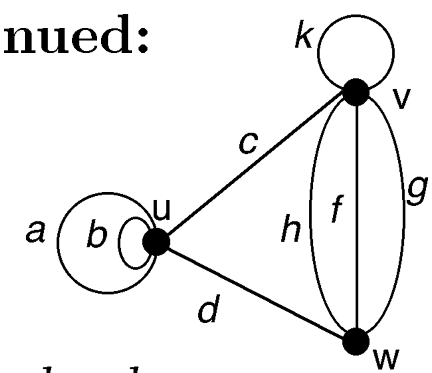

$$V = \{u,\ v,\ w\}\ and\ E = \{a,\ b,\ c,\ d,\ f,\ g,\ h,\ k\}$$

| edge | $a$ | $b$ | $c$ | $d$ | $f$ | $g$ | $h$ | $k$ |
|---|---|---|---|---|---|---|---|---|
| endpts | $u$ | $u$ | $u$ | $w$ | $v$ | $v$ | $w$ | $v$ |
| | $u$ | $u$ | $v$ | $u$ | $w$ | $w$ | $v$ | $v$ |

# Graph Representation (incidence matrix)

- *Incidence matrix* representation: rows indexed by each vertex, columns indexed by each edge. Degrees of graph are row sums.

**Example 10.3.1, continued:**



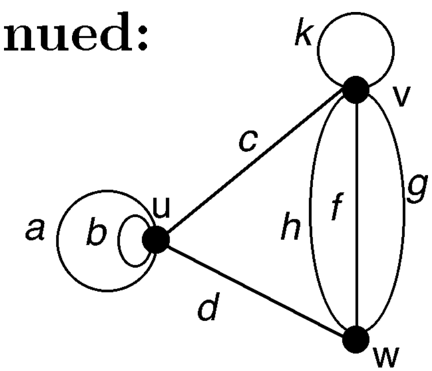

| | $a$ | $b$ | $c$ | $d$ | $f$ | $g$ | $h$ | $k$ |
|---|---|---|---|---|---|---|---|---|
| $u.$ | 2 | 2 | 1 | 1 | 0 | 0 | 0 | 0 |
| $v.$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 2 |
| $w.$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |

# Graph Representation (adjacency list)

- *Adjacency list* representation: table of the adjacency lists for each vertex ***v*** of the graph. The list contains each vertex ***w*** of G once for each edge between {***v***, ***w***}.

**Example 10.3.1, continued:**

| | | | | | |
|---|---|---|---|---|---|
| $u.$ | $u$ | $u$ | $v$ | $w$ | |
| $v.$ | $u$ | $v$ | $w$ | $w$ | $w$ |
| $w.$ | $u$ | $v$ | $v$ | $v$ | |



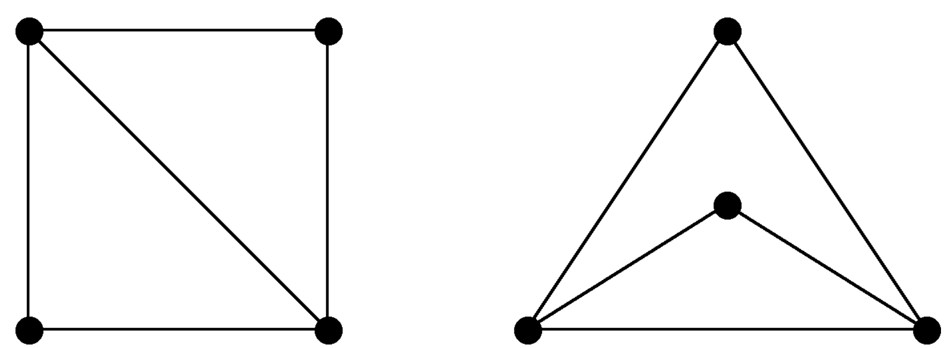

| | $u$ | $v$ | $w$ |
|---|---|---|---|
| $u.$ | 2 | 1 | 1 |
| $v.$ | 1 | 1 | 3 |
| $w.$ | 1 | 3 | 0 |

# Graph Isomorphism

- Definition: two graphs {G,H} are *isomorphic* ("same-form") if there exists a one-to-one and onto function

$$f : V_G \rightarrow V_H$$

such that $\forall u, v \in V_G$ the # of edges between $f(u)$ and $f(v)$ equals the # of edges between u and v.

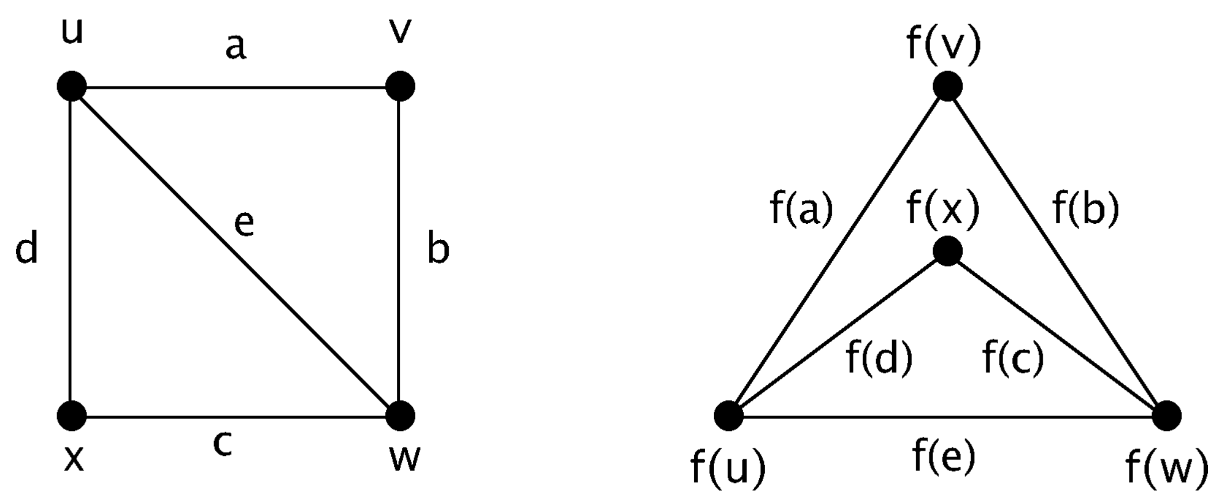


# Simple Isomorphism

- Proposition: two simple graphs {G,H} are *isomorphic* IFF there exists a bijection

$$f: V_G \rightarrow \mathbf{V}_H$$

such that vertices $f$(u) and $f$(v) are adjacent in H, IFF vertices u and v are adjacent in G.



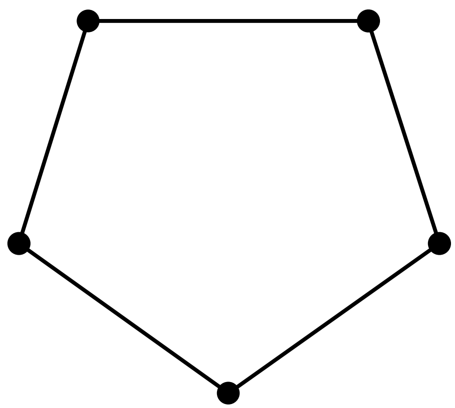
# Preserved Under Isomorphism

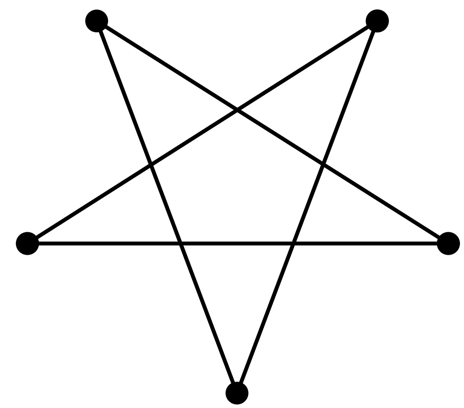
- A property of a graph is said to be *preserved under isomorphism* if whenever G has that property, every graph isomorphic to G also has that property.
- Graph *invariants*:
  1. Number of vertices (=)
  2. Number of edges (=)
  3. Degree sequence (=)

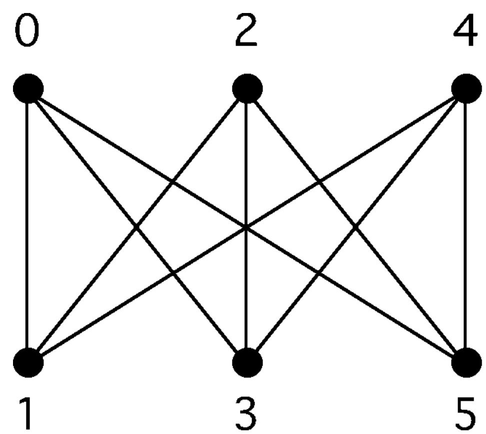
  \+ "Connection properties"

# Non-Isomorphic Graphs (examples)

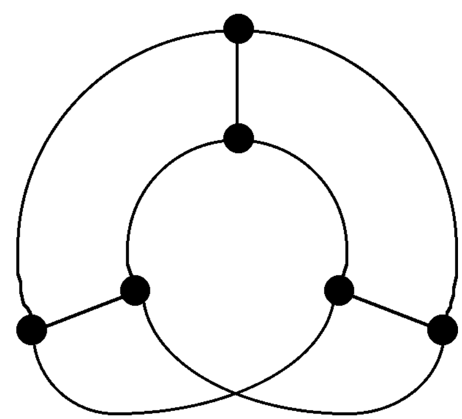
# Isomorphic Graphs (examples)





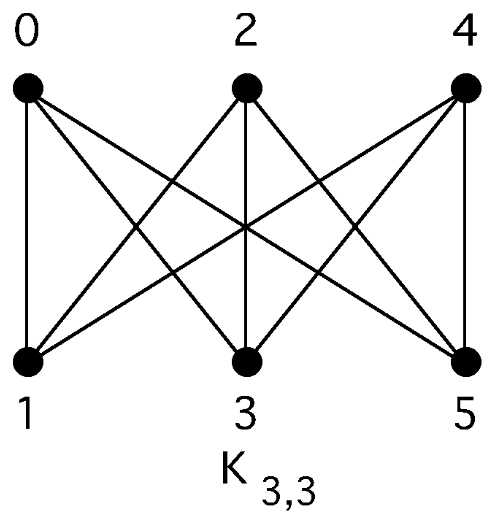


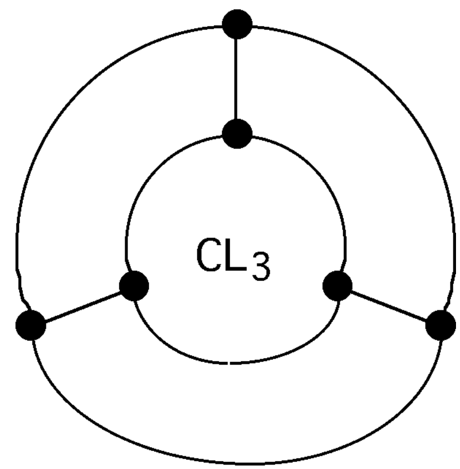

$K_{3,3}$

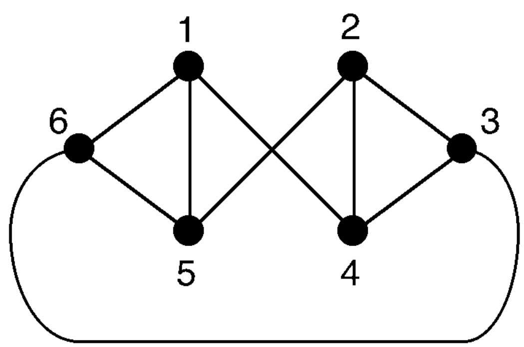


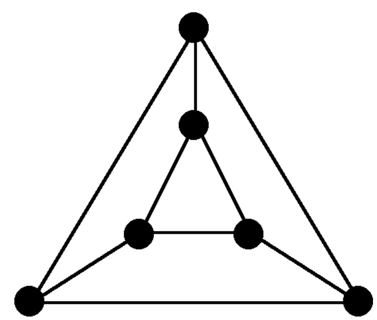
$ML_3$

# Isomorphic or not? (test yourself)

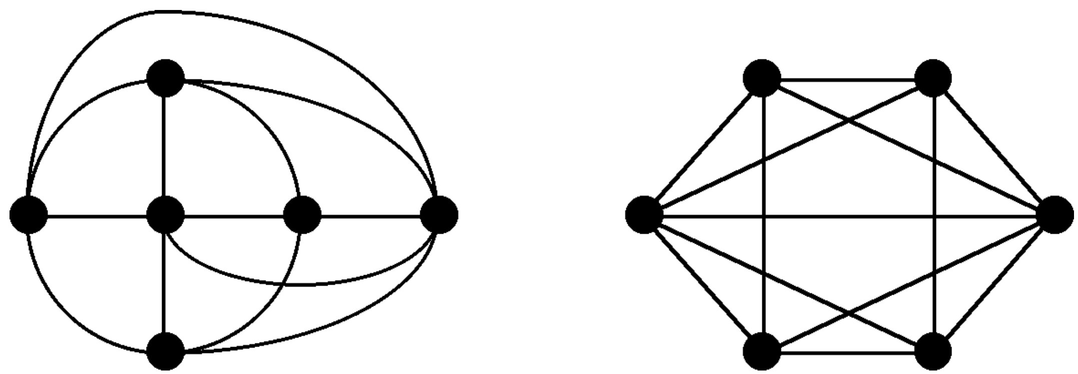


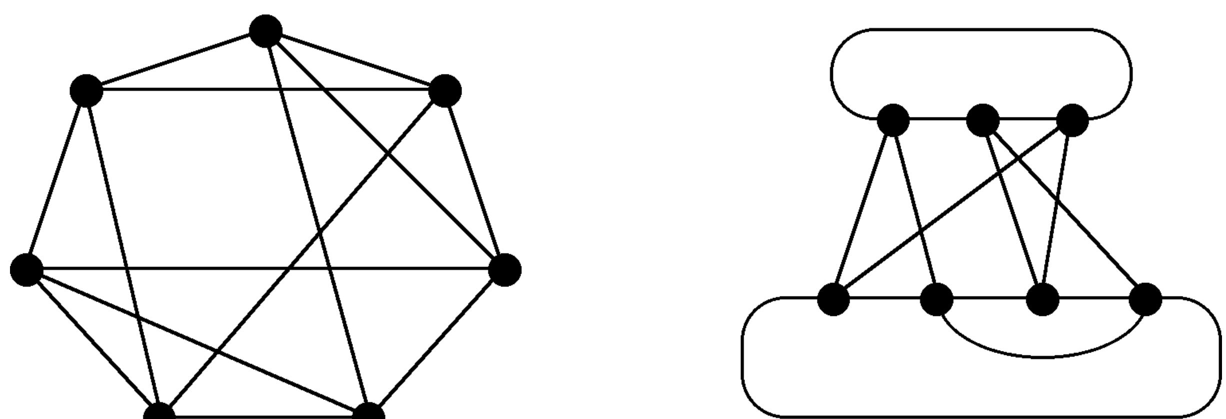



# Isomorphic or not? (test yourself)





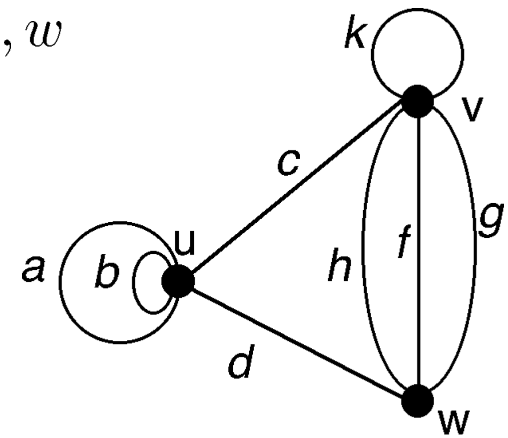
# Walks

- Definition: a *walk* from vertex $v_0$ to vertex $v_n$ is an alternating sequence of vertices and the edges joining them: *W* = $v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$
- The walk consists of two external vertices: *initial* ($v_0$) and *final* ($v_n$), and a set of *internal* vertices.
- The walk is *closed* if $v_0 = v_n$
- The *length* of a walk = # of edge steps.

# Walks (example)

**Example 10.4.1:** Consider three walks:

$$W_1 = u, c, v, f, w, h, v, f, w$$
$$W_2 = v, f, w, h, v$$
$$W_3 = w, f, v, h, w$$




- Observe:
  1. Length($W_1$) = 4
  2. Represent $W_1$ unambiguously: {c, f, h, f}
  3. Represent $W_2$ ambiguously: {v, w, v}
  4. Represent $W_2$ ambiguously: {f, h}

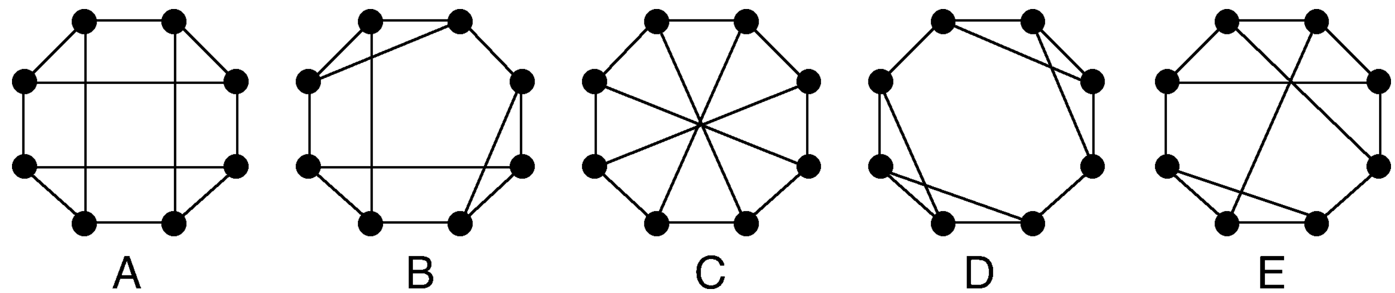
# Paths

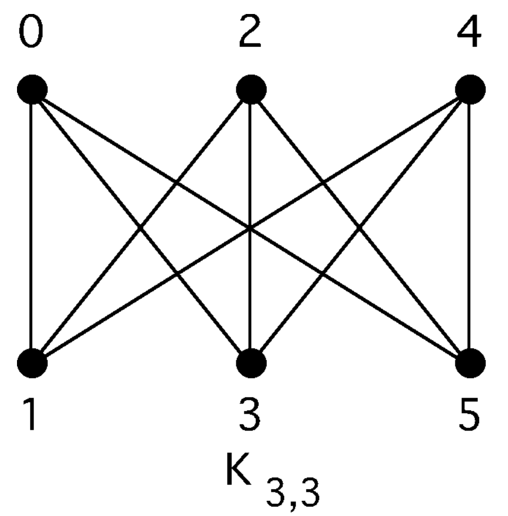
- Definition: a *path (trail)* is a walk with no repeated edges.
- Definition: a *cycle (closed path)* is a path in which the only repeated vertex is the external one (initial/final).
- Remark: paths and cycles should not be confused with “path graphs” and “cycle graphs”.
- Additional terminology: “open path”, “simple path”

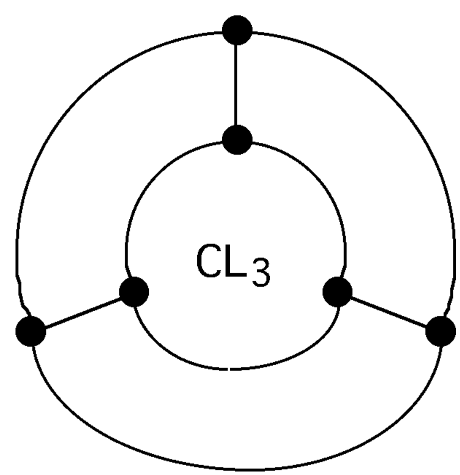
# Cycles & Isomorphism

- Proposition: suppose we have an isomoprhism $f$ between two graphs {G,H}, and let W be a walk in G. Then, $f$(W) is a walk in H.
  - $f: G \rightarrow \mathbf{H}$
  - $W = v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n$
  - $f(W) = f(v_0), f(e_1), f(v_1), \ldots, f(v_{n-1}), f(e_n), f(v_n)$
- Proposition: suppose we have an isomoprhism $f$ between two graphs {G,H}, and let C be a k-cycle in G. Then, $f$(C) is a k-cycle in H.

# Non-isomorphic Graphs (cycles)

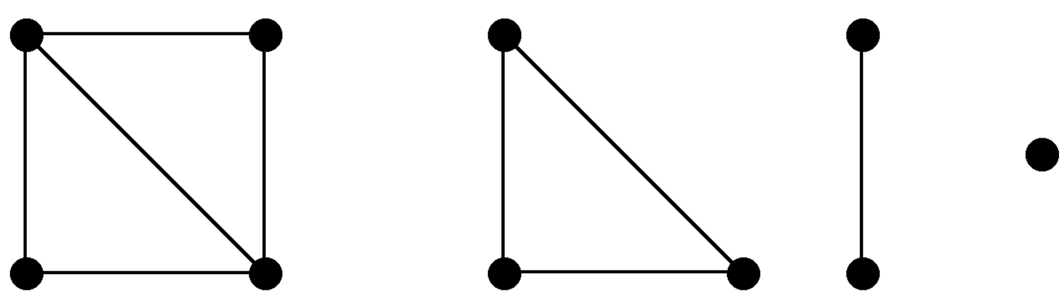


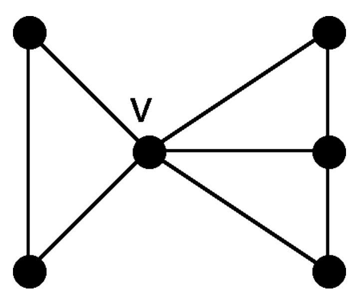
# Connectedness

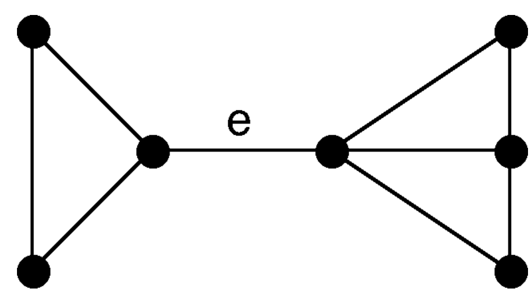
- Definition: a graph G is *connected* if there exists a path $\forall u, v \in V_G$
- Definition: a *component* of a graph G is a *maximal connected subgraph*. The subgraph is not properly contained in any larger connected subgraph.
- Definition: a digraph D is *strongly connected* if there exists a directed path $\forall u, v \in V_D$



# Cutpoints & Cutedges

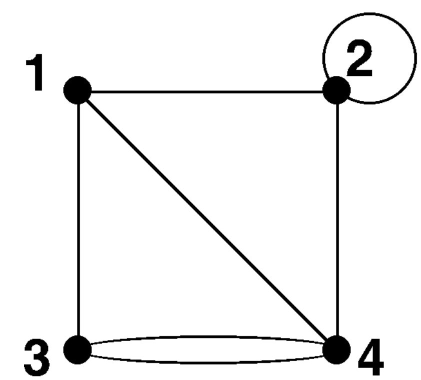
- Definition: a *cutpoint* of a graph is a vertex whose removal increases the number of components.
- Definition: a *cutedge* of a graph is an edge whose removal increases the number of components.



# Naive Connectedness Test

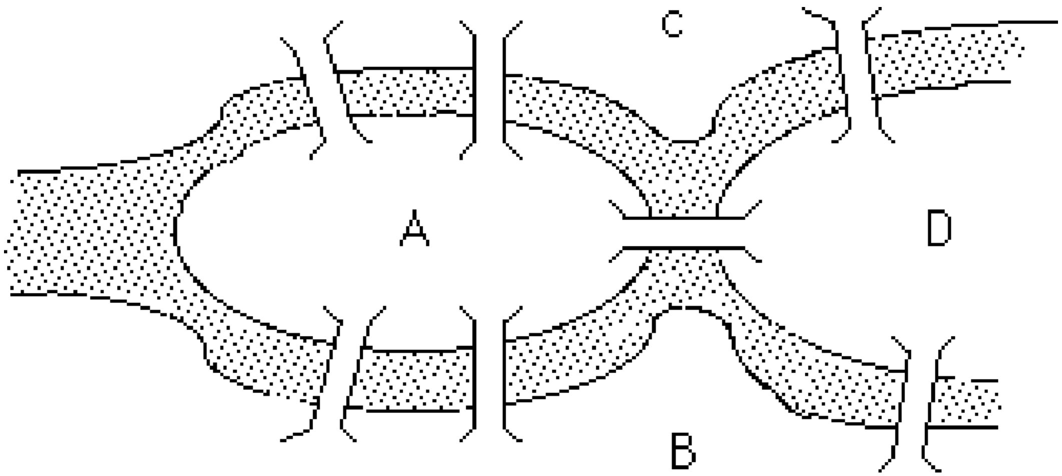
- Proposition: let *A* be the adjacency matrix of graph *G* (with n vertices). Then, $A^n[i, j]$ is the number of walks of length n between vertices *{i, j}*.
- Connectedness Test for G:
  1. Calculate $A^2 \ldots A^{n-1}$
  2. Verify that no entry remained zero throughout.

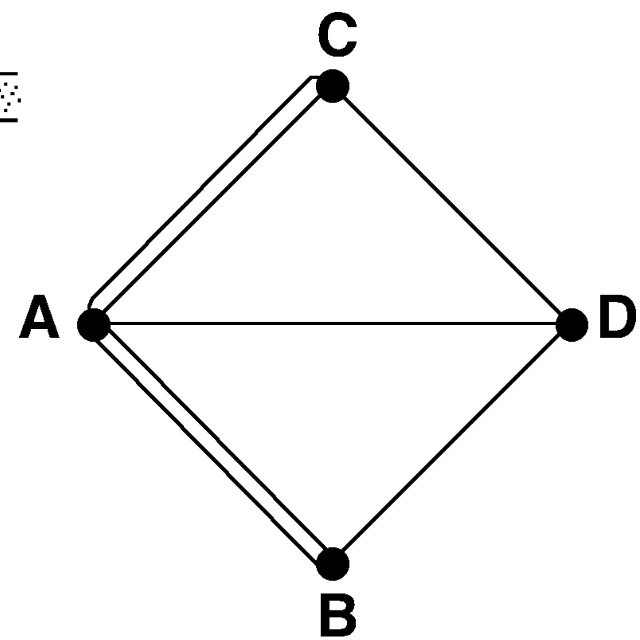



| A | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 |
| 2 | 1 | 2 | 0 | 1 |
| 3 | 1 | 0 | 0 | 2 |
| 4 | 1 | 1 | 2 | 0 |

| $A^2$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 3 | 3 | 2 | 3 |
| 2 | 3 | 6 | 3 | 3 |
| 3 | 3 | 3 | 5 | 1 |
| 4 | 3 | 3 | 1 | 6 |

# Konigsberg Bridge Problem



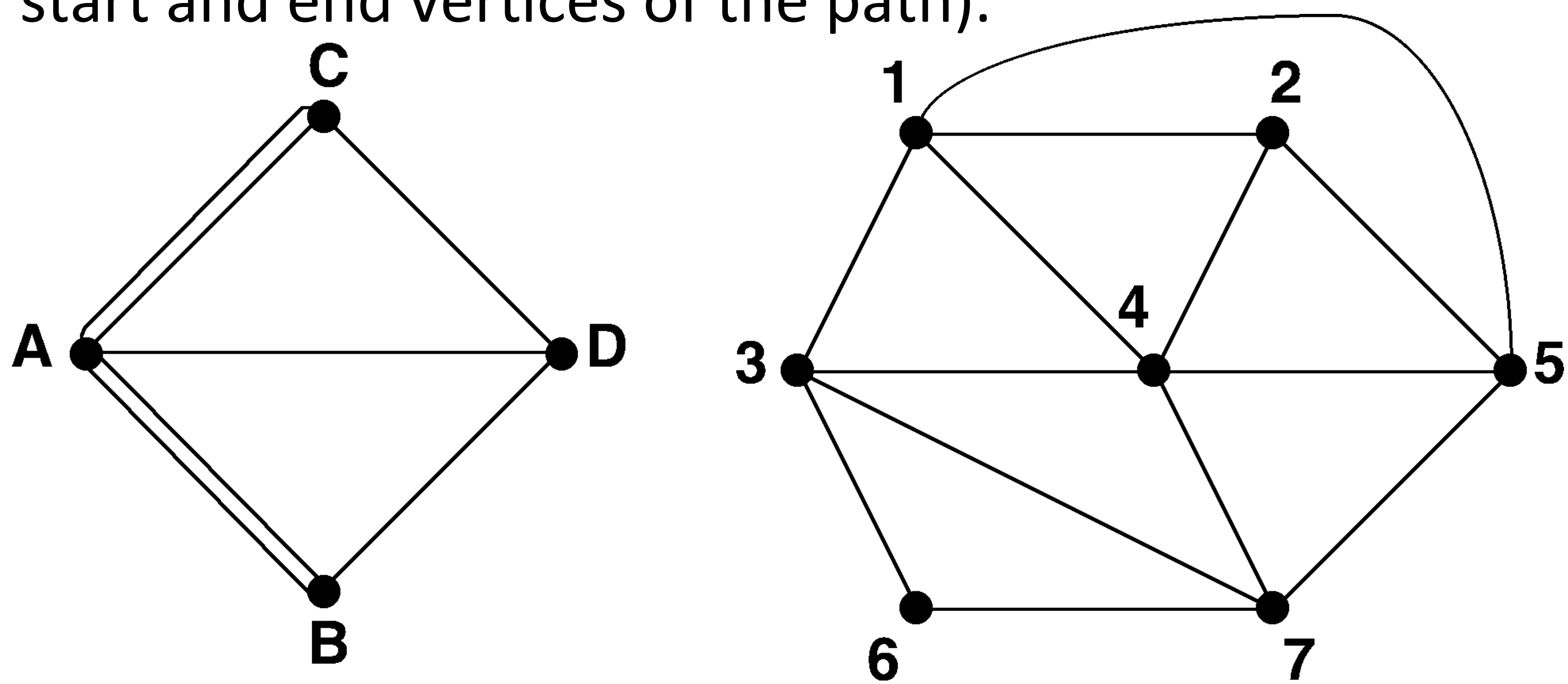



# Eulerian Tour

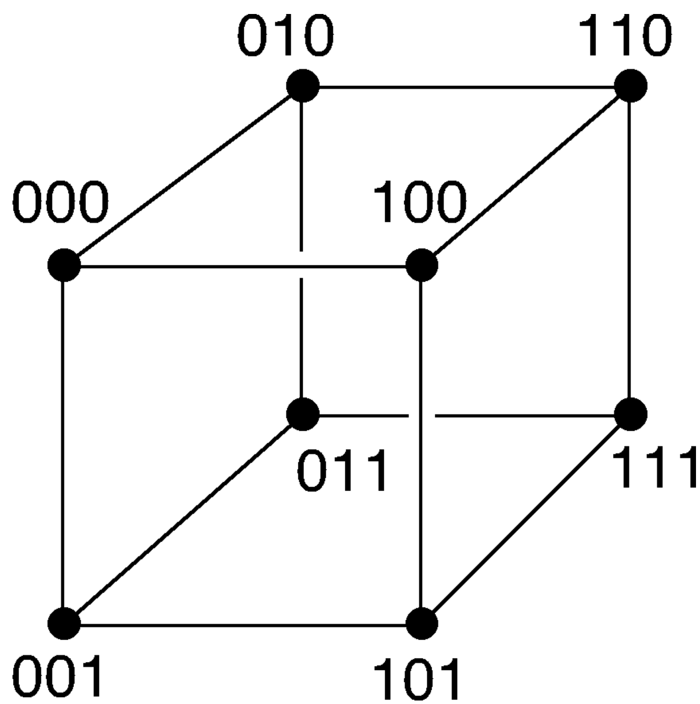
- Definition: a *Eulerian path* in a graph is a path that traverses *every edge* exactly once.
- Definition: a *Eulerian tour (circuit, cycle)* in a graph is a Eulerian path that starts and ends with the same vertex.
- Definition: a *Eulerian graph* is a graph that has a Eulerian tour.

# Eulerian Graph

- Theorem: a connected graph is Eulerian IFF every vertex has even degree.
- Theorem: a connected graph has a Eulerian path IFF it has exactly two vertices of odd degree (these are the start and end vertices of the path).

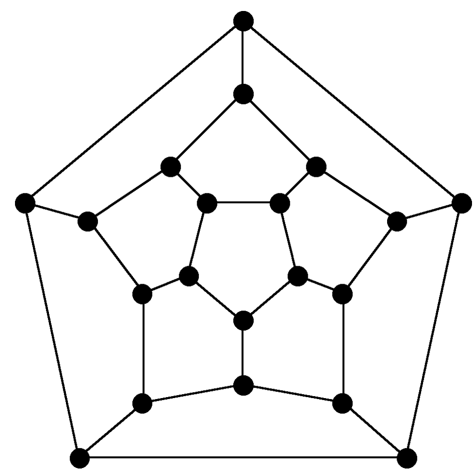


# Gray Code Problem



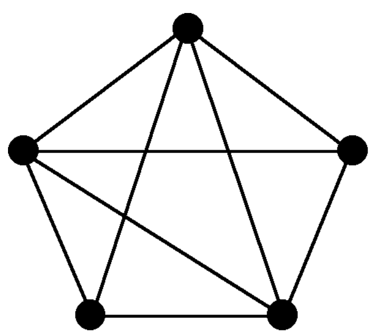
# Hamiltonian Tour

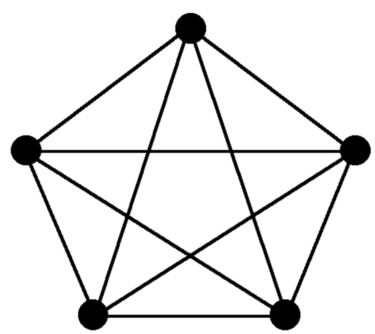
- Definition: a *Hamiltonian path* in a graph is a path that visits *every vertex* exactly once.
- Definition: a *Hamiltonian tour (circuit, cycle)* in a graph is a Hamiltonian path that starts and ends with the same vertex.
- Definition: a *Hamiltonian graph* is a graph that has a Hamiltonian tour.

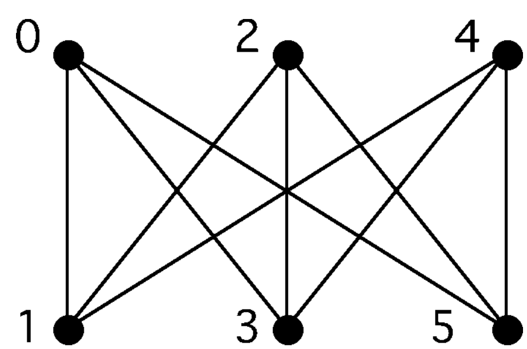


# Planar Graphs

- Definition: a graph is *planar* if it can be drawn without edge crossings in the plane.
- *Imbedding problem*: given a graph G and a surface S, is it possible to draw G on S without any edge-crossings?
- *Planarity problem*: imbedding problem where S is the sphere or the plane.


$K_5$-e: planar

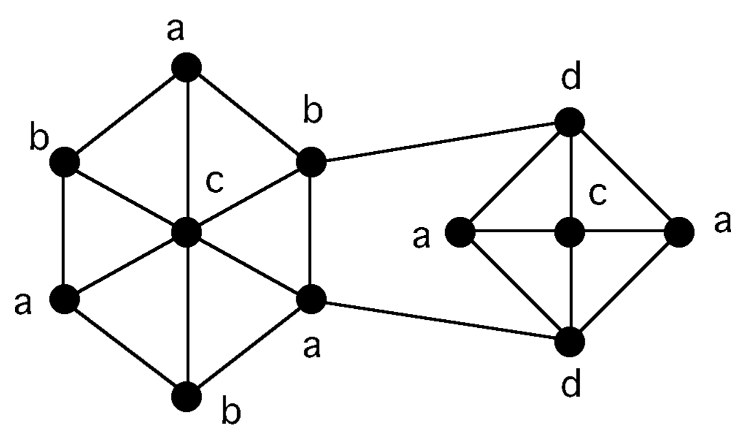

$K_5$: non-planar



$K_{3,3}$ :non-planar

# Graph Coloring

- Definition: an *n-coloring* of a graph G is a function from its vertex set $V_G$ onto the set {1,2,...,n} whose elements we regard as "colors".
- Definition: an *n-coloring* is *proper* if no pair of adjacent vertices gets the same color.
- Definition: a graph is *n-colorable* if it has a proper n-coloring.



# Chromatic Number

- Definition: a *chromatic number* of a graph G is the minimum number of colors for which the graph is colorable:

  $\chi(G) = \min \{n \in Z^+$ such that G is n-colorable$\}$

- Definition: a graph is *n-chromatic* if $\chi(G) = n$.
- Example: a graph that contains $K_3$ requires at least 3 colors.