W3203
Discrete Mathematics

Counting

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Outline

- Bijection rule
- Sum, product, division rules
- Permutations and combinations
- Sequences with (restricted) repetitions
- The Pigeonhole Principle
- Inclusion - Exclusion
- Binomial Theorem, Pascal's Identity
- Text: Rosen 6.1 – 6.5
- Text: Lehman 14

The Bijection Rule

- Principle: count one thing by counting another
- *Bijection rule*: find a bijection between two sets, A & B. Then the sets have the same size.
- General strategy: get really good at counting just a few things, then use bijections to count everything else!
- Example:
  - Set A: all ways to select a dozen donuts when five varieties are available
  - Set B: all 16-bit sequences with exactly 4 ones
  - Map donuts to to sequences of bits
  - Proves sets have same size, without knowing how big exactly!

The Sum Rule

- Number of objects in the whole equals the sum of objects in the (disjoint) parts
- *Sum rule*: let A and B be finite disjoint sets ($A \cap B = \emptyset$). Then $|A \cup B| = |A| + |B|$
- Can we generalize the rule to n sets?
- If $A_1, A_2, \ldots, A_n$ are *pairwise disjoint* sets, then:

  $|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$
- What if the sets overlap? Inclusion-Exclusion

Sum Rule (examples)

- Example 1: suppose there are
  - 19 French speakers, 17 English speakers and no bilingual speakers
  - How many ways are there to choose someone who speaks either language?
  - Answer: 19 + 17 = 36
- Example 2: suppose there are
  - 20 French speakers, 40 English speakers, 60 Russian speakers, and 80 Spanish speakers, but no bilingual speakers among them
  - How many ways are there to choose someone who speaks one of these languages?
  - Answer: 20 + 40 + 60 + 80 = 200

The Subtraction Rule

- If two sets (parts) overlap, then we count some objects twice when we count the whole
- *Rule*: let A and B be finite sets. Then,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- General principle (n sets): *inclusion-exclusion*

Inclusion-Exclusion

- Recall: if two sets overlap, then we count some objects twice when we count the whole
- What about three overlapping sets:
- *Rule*: let A,B,C be finite sets. Then,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

- General principle (n sets): sums with alternating signs, the sum of the k-way intersections getting the sign $(-1)^{k-1}$

The Product Rule

- How can we count sequences?
- *Product rule*: let A and B be sets. Then the set of all sequences whose 1st term is from A, and 2nd term is from B is their Cartesian product $A \times B$. If sets are finite, then: $|A \times B| = |A| \cdot |B|$
- Example:
  - Suppose there are 19 CS majors and 17 math majors
  - Count the # of ways to pick two students with different majors
  - Answer: 19 x 17 = 323

Iterated Product Rule

- Can we generalize the product rule to n sets?
- *Iterated Product rule*: if $A_1, A_2, \ldots, A_n$ are finite sets, then:

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

$$\left| \prod_{j=1}^{n} A_j \right| = \prod_{j=1}^{n} |A_j|$$

- Example:
  - Passwords on a given system consist of n characters: small case letters and digits
  - Count the # of valid passwords
  - Answer: (26+10) x (26+10) x ... x (26+10) = 36^n

Counting Passwords Example

- Example: passwords on a given system can be 6-8 characters longs, where each character is a lowercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
- Solution: combine counting rules
  - Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8
  - By Sum Rule: $P = P_6 + P_7 + P_8$
  - To find each of P_6, P_7, and P_8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters

Counting Passwords Example (solution)

- How many possible passwords are there?
- Solution: combine counting rules
  - Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.
  - By Sum Rule: $P = P_6 + P_7 + P_8$
  - By Product Rule:

    $P_6 = 36^6 - 26^6$

    $P_7 = 36^7 - 26^7$

    $P_8 = 36^8 - 26^8$

    $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$

Generalized Product Rule

Rule 14.3.1 (Generalized Product Rule)**.** *Let S be a set of length-k sequences. If there are:*

- *n_1 possible first entries,*
- *n_2 possible second entries for each first entry,*

  $\vdots$

- *n_k possible kth entries for each sequence of first $k-1$ entries,*

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

Permutations

- Definition: a *permutation* of a set S is a sequence that contains every element of S exactly once. It is a bijection from a set onto itself
- How many permutations of an n-element set are there? Answer: $n \cdot (n-1) \cdot (n-2) \ldots \cdot 2 \cdot 1 = n!$
- *Ordered r-selection* (r-permutation) from a set S (without repetition) is a sequence of r objects from S.
- Notation: $n^{\underline{r}} = n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$

$$P(n,r) = \frac{n!}{(n-r)!}$$

Permutations (examples)

- Notation: $n^{\underline{r}} = n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$
- Example 1:
  - Given a standard 52-card deck
  - Count the # of ways to deal a 5-card sequence
  - Answer: $52 \times 51 \times 50 \times 49 \times 48 = 52^{\underline{5}}$
- Example 2:
  - Given a starting location and 7 cities to visit
  - In how many ways (orders) can you visit these cities?
  - Answer: $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

The Division Rule

- A *k-to-1 function* maps exactly k elements of the domain to every element of the codomain
  - Example: the function mapping each ear to its owner is 2-to-1
- If the finite set A is the union of n pairwise-disjoint subsets, each with d elements, then **n = |*A*| / *d***

Rule 14.4.1 (Division Rule). *If* $f : A \to B$ *is k-to-1, then* $|A| = k \cdot |B|$.

Division Rule (round table example)

- Example: how many ways are there to seat 4 people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?
  1. Number the seats around the table from 1 to 4 proceeding clockwise.
  2. There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4
  3. Thus there are $4! = 24$ ways to order the four people
  4. But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating
  5. Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.

The Subset Rule

- How many r-element subsets of an n-element set are there?
- *Subset rule*: the number of k-element subsets of an n-element set is "n choose k"
- *Unordered k-selection* (k-combination) from a set S is a subset of k objects from S.
- Notation:

$$\binom{n}{k} = C(n,k) = \frac{n^{\underline{k}}}{k!} = \frac{n!}{(n-k)!k!}$$

Subset Rule (derivation)

- How can we count subsets?
- Given set with n elements,
  1. Construct mapping from each permutation into a k-element subset by taking the first ***k*** elements of the permutation
  2. There are ***k!*** possible permutations of the first k elements
  3. There are ***(n – k)!*** permutations of the remaining elements
  4. By *Product rule*, there are exactly ***k! (n – k)!*** permutations of the set that map to a particular subset
  5. Constructed mapping which is ***k! (n – k)! – to – 1***
  6. There are ***n!*** permutations of an n-element set
  7. By *Division rule*, ***n! = k! (n – k)! C(n, k)*** $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$

Subset Rule (examples)

- Notation: C(n, k)

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

- Example 1:
  - Given a standard 52-card deck
  - Count the # of 5-card hands that can be dealt
  - Answer: C(52, 5) = $52^{\underline{5}}$ / 5!
- Example 2:
  - Count the number of n-bit sequences with exactly k ones
  - Answer: $\mathrm{C}(\mathrm{n}, \mathrm{k})$
- Example 3:
  - How many ways to select n donuts with exactly k varieties?
  - Answer: $\mathrm{C}(\mathrm{n}+(\mathrm{k}\text{-}1), \mathrm{n})$

Sequences of Subsets

- Choosing a k-element subset of an n-element set is the same as splitting the set into two subsets (size k, size n-k)
- Generalization to more than two subsets: "sequence with restricted repetitions", "permutation with indistinguishable objects"
- Let A be an n-element set and $(k_1, k_2, \ldots, k_m)$ be nonnegative integers whose sum is n. Define a *split of* A as a sequence $(A_1, A_2, \ldots, A_m)$ of disjoint subsets, $|A_i| = k_i$
- To count number of splits, follow approach used to derive subset rule

Multinomial Coefficient

Definition 14.6.1. For $n, k_1, \ldots, k_m \in \mathbb{N}$, such that $k_1 + k_2 + \cdots + k_m = n$, define the *multinomial coefficient*

$$\binom{n}{k_1, k_2, \ldots, k_m} ::= \frac{n!}{k_1! k_2! \ldots k_m!}.$$

Rule 14.6.2 (Subset Split Rule)**.** *The number of* $(k_1, k_2, \ldots, k_m)$*-splits of an* n*-element set is*

$$\binom{n}{k_1, \ldots, k_m}.$$

Sequences of Subsets (examples)

- Notation: $\dfrac{n!}{k_1!\,k_2!\,\ldots k_m!}$
- Example 1:
  - Count the # of strings obtainable by rearranging letters of BANANA
  - Answer: 3A 1B 2N ➔ $\dfrac{6!}{3!1!2!}$
- Example 2:
  - Count the # of strings obtainable by rearranging letters of BOOKKEEPER
  - Answer: 1B 3E 2K 2O 1P 1R ➔ $\dfrac{10!}{1!3!2!2!1!1!}$

Binomial Coefficients

- Definition: a *binomial* is a sum of two terms, a + b
- Definition: the *binomial coefficients* are the coefficients of the terms in the expansion of the binomial to some power $(a + b)^n$

$$
\begin{aligned}
(a+b)^4 = \quad & aaaa + aaab + aaba + aabb \\
+ \; & abaa + abab + abba + abbb \\
+ \; & baaa + baab + baba + babb \\
+ \; & bbaa + bbab + bbba + bbbb
\end{aligned}
$$

$$
(a+b)^4 = \binom{4}{0} \cdot a^4 b^0 + \binom{4}{1} \cdot a^3 b^1 + \binom{4}{2} \cdot a^2 b^2 + \binom{4}{3} \cdot a^1 b^3 + \binom{4}{4} \cdot a^0 b^4
$$

Binomial Theorem

- General statement for expansions
  - Extends to multinomials
  - Explains why n-choose-k is called a binomial coefficient

Theorem 14.6.4 (Binomial Theorem)**.** *For all* $n \in \mathbb{N}$ *and* $a, b \in \mathbb{R}$*:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

The Pigeonhole Principle

- The problem: a drawer in a dark room contains red, green, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?
- *The Pigeonhole Principle*: if there are more pigeons than holes they occupy, then at least two pigeons must be in the same hole
- If k + 1 objects are placed into k boxes, then there is at least one box containing two or more objects

Pigeonhole Principle (approach)

1. The set A (pigeons, objects)
2. The set B (pigeonholes, boxes)
3. The function f (rule for mapping pigeons to holes)

Rule 14.8.1 (Pigeonhole Principle). *If* $|A| > |B|$, *then for every total function* $f : A \to B$, *there exist two different elements of* A *that are mapped by* f *to the same element of* B.

Set-Theoretic Pigeonhole Principle

- Consider a function $f : A \to B$ with finite domain and codomain. If any two of three properties hold, then so does the third:
  1. *f is one-to-one*
  2. *f is onto*
  3. $|A| = |B|$
- Example: given 7 people, suppose no two of them were born on same week day. Then, one each day, someone was definitely born

Pigeonhole Principle (examples)

- Function $f : A \to B$ with finite domain and codomain:
  1. f *is one-to-one*
  2. f *is onto*
  3. $|A| = |B|$
- Example: given n people, suppose no two of them were born on same week day, and at least one was born on each day. Then, n = 7
- Example: Boston has about 500,000 non-bald people, and say the number of hairs on a person's head is at most 200,000. We can conclude that at least two people in Boston have exactly the same number of hairs. We don't know who they are, but we know they exist!

Generalized Pigeonhole Principle

- *Generalized Pigeonhole Principle:* if $|A| > k \cdot |B|$, then every function $f : A \to B$ with finite domain and codomain maps at least k+1 different elements of A to the same element of B
- Example: A = 500,000 non-bald people, B = 200,000 hairs. Since $|A| > 2|B|$, at least 3 people in Boston have exactly the same number of hairs. We still don't know who they are, but we know they exist!

Symmetry Identity

- How do you prove the identity?

$$\binom{n}{k} = \binom{n}{n-k}$$

- You have n objects but want to keep only k of them
- You can select k objects to keep
- You can select n-k objects to throw out

Combinatorial Proof

- We wish to prove that a given equality holds
- *Combinatorial proof*: an argument that establishes and algebraic fact by relying on counting principles
- General proof structure:
  1. Define a set S
  2. Show that |S| = n by counting one way
  3. Show that |S| = m by counting another way
  4. Conclude that n = m
- Caveat: it can be tricky to define the set S

Pascal's Identity

- Suppose you are one of ***n*** people vying for ***k*** spots on a team. In how many ways can the team be selected?
- Approach 1: count # of k-element subsets of an n-element set
- Approach 2: consider two cases
  1. You are selected: count # of k-1 element subsets of an n-1 element set
  2. You are not: count # of k element subsets of an n element set
  3. By sum rule: case 1 + case 2

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Identity (algebraic proof)

$$
\begin{aligned}
\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(n-r)!\,(r-1)!} + \frac{(n-1)!}{(n-r-1)!\,r!} \\
&= \frac{(n-1)!\,r}{(n-r)!\,r!} + \frac{(n-1)!\,(n-r)}{(n-r)\,!r!} \\
&= \frac{(n-1)!\,[r+(n-r)]}{(n-r)!\,r!} \\
&= \frac{(n-1)!\,n}{(n-r)!\,r!} \\
&= \frac{n!}{(n-r)!\,r!} = \binom{n}{r}
\end{aligned}
$$

♢

Pascal's Triangle

	r = 0	1	2	3	4	5	6
n = 0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Often Pascal's triangle appears this way:

```
n = 0                    1
  1                  1       1
  2              1       2       1
  3          1       3       3       1
  4      1       4       6       4       1
  5  1       5       10      10      5       1
  6  1   6       15      20      15      6       1
```

BINOMIAL COEFFICIENT IDENTITIES

Theorem 6.4.5. *Subset size sum.*

$$\sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Pf: (1st proof: computational)
By induction on n, using Pascal's recursion. ♢

Pf: (2nd proof: combinatorial)
Both sides count all the subsets. ♢

Pf: (3rd proof: corollary to binomial thm)
Expand $(x+y)^n$ with $x = y = 1$. ♢

Theorem 6.4.6. *Alt sum of binom coeffs.*

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$$

Pf: Expand $(x+y)^n$ with $x = 1$ and $y = -1$. ♢

The set $\{1,2,3,4\}$ has 16 subsets.

$\varnothing$						$\binom{4}{0}$
1	2	3	4			$\binom{4}{1}$
12	13	14	23	24	34	$\binom{4}{2}$
123	124	134	234			$\binom{4}{3}$
1234						$\binom{4}{4}$
						$2^4 = 16$

The set $\{1,2,3,4\}$ has 15 partitions, as follows:

1234						
12|34	13|24	14|23	1|234	2|134	3|124	4|123
1|2|34	1|3|24	1|4|23	2|3|14	2|4|13	3|4|12	
1|2|3|4						

The types of these partitions are

$$4 \quad 22 \quad 13 \quad 112 \quad 1111$$

corresponding to the partitions of the number 4.

Partitions

- Objective: count the partitions of a set of n objects without listing them all
- Definition: the *Stirling subset number* (Stirling coefficient of the 2nd kind) is the # of partitions of n objects into r unlabeled cells $\left\{ {n \atop r} \right\}$
- "Distinguishable objects in undistinguishable boxes"

$$\left\{ {4 \atop 1} \right\} = 1 \quad \left\{ {4 \atop 2} \right\} = 7 \quad \left\{ {4 \atop 3} \right\} = 6 \quad \left\{ {4 \atop 4} \right\} = 1$$

Stirling's Recursion

$$\left\{ {n \atop r} \right\} = \left\{ {n-1 \atop r-1} \right\} + r \left\{ {n-1 \atop r} \right\}$$

Pf: The n^{th} object is isolated in a cell by itself in

$$\left\{ {n-1 \atop r-1} \right\}$$

partitions. Each of the remaining partitions is formed by first partitioning the $n-1$ other objects into r nonempty cells and then selecting one of them as a cell for the n^{th} object. By the rule of product, there are

$$r \left\{ {n-1 \atop r} \right\}$$

ways to do this. The rule of sum now implies the conclusion. ◇