Math S1201
Calculus 3
Chapter 14.8

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Outline

- CH 14.8 Lagrange Multipliers
  - Maximizing / minimizing function subject to equality constraint
  - General approach for any dimension
  - Geometrical interpretation and proof
  - Method of Lagrange Multipliers
  - Two equality constraints

Guiding Eyes (14.8)

A. How do you find the extrema of a function subject to an equality constraint?

B. How do you justify the Method of Lagrange Multipliers?

C. How do you find the extrema of a function subject to *multiple* equality constraints?

Extrema of function subject to one equality constraint

Consider functions of two variables: ***f(x,y)***

We wish to find the extrema of *f* where (x,y) is restricted to lie on the level curve ***g(x,y) = k***.

$$\max_{x,y} f(x,y) \quad \text{subject to: } g(x,y) = k$$

Formally stated, we have a more general problem:

$$\max_{\boldsymbol{x}} f(\boldsymbol{x}) \quad \text{s.t. } g(\boldsymbol{x}) = 0$$

Approach for functions of two variables:

1) Use the constraint ***g(x,y) = 0*** to exress **y** in terms of **x**: **y = h(x)**
2) Substitute **y = h(x)** into ***f(x,y)*** and obtain a function of a single variable.
3) Differentiate to find critical points ($\mathbf{x}^*$), verify extrema.
4) Substitute back to get $\mathbf{y}^* = \mathbf{h}(\mathbf{x}^*)$.

Q. Would this approach work for functions of three variables?

Q. Would this approach work for *any* function *f* and constraint *g*?

What is the distance between a point and a plane?

Problem: find the shortest distance from the point $P(p_1,p_2,p_3)$ to the plane ***ax + by + cz + d = 0***.

Solution: Recall, we solved this problem
$$D = \frac{|ap_1 + bp_2 + cp_3 + d|}{\sqrt{a^2 + b^2 + c^2}}$$
Consider this as an optimization problem:

$$f(x,y,z) = \sqrt{(x - p_1)^2 + (y - p_2)^2 + (z - p_3)^2}$$

$$g(x,y,z) = ax + by + cz + d$$

$$\min_{x,y,z} f(x,y,z) \quad \text{s.t. } g(x,y,z) = 0$$

1) Use the constraint ***g(x,y,z) = 0*** to exress **z** in terms of **x,y**: **z = h(x,y)**
2) Substitute **z = h(x,y)** into ***f(x,y,z)*** and obtain a function of two variables.
3) Compute partial derivatives to find critical points ($\mathbf{x^*,y^*}$), verify extrema.
4) Substitute back to get $\mathbf{z^* = h(x^*, y^*)}$.

What is the distance between a point and a plane?

Problem: find shortest distance from $P(p_1,p_2,p_3)$ to $\boldsymbol{ax + by + cz + d = 0}$.

$$f(x,y,z) = \sqrt{(x-p_1)^2 + (y-p_2)^2 + (z-p_3)^2}$$

$$g(x,y,z) = ax + by + cz + d$$

$$\min_{x,y,z} f(x,y,z) \quad \text{s.t. } g(x,y,z) = 0$$

$$1)\ z = (-ax - by - d)/c$$

$$2)\ f(x,y) = (x-p_1)^2 + (y-p_2)^2 + \left(\frac{-ax - by - d - p_3}{c}\right)^2$$

$$3)\ f_x = 2(x-p_1) - 2a\left(\frac{-ax - by - d - p_3}{c}\right) = 0$$

$$f_y = 2(y-p_2) - 2b\left(\frac{-ax - by - d - p_3}{c}\right) = 0$$

$$\Rightarrow x = x^*, y = y^*$$

$$4)\ z^* = \frac{-ax^* - by^* - d}{c} \Rightarrow d = f(x^*, y^*, z^*)$$

What is the maximum volume of a box?

Problem: find the max volume of a box without lid made from T(m^2) amount of material.

Solution: consider this as an optimization problem, where x,y,z are the dimensions of the box.

$$V(x,y,z) = xyz \qquad g(x,y,z) = 2xz + 2yz + xy - T$$

$$\max_{x,y,z} f(x,y,z) \quad \text{s.t. } g(x,y,z) = 0$$

$$z = \frac{T - xy}{2(x+y)} \Rightarrow V = xy\left(\frac{T - xy}{2(x+y)}\right) = \frac{Txy - x^2y^2}{2(x+y)}$$

$$\frac{\partial V}{\partial x} = \frac{y^2(T - 2xy - x^2)}{2(x+y)^2} \quad \frac{\partial V}{\partial y} = \frac{x^2(T - 2xy - y^2)}{2(x+y)^2}$$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0 \quad x \neq 0, y \neq 0 \Rightarrow T - 2xy - x^2 = 0, T - 2xy - y^2 = 0$$

$$\Rightarrow x^2 = y^2, x > 0, y > 0 \Rightarrow x = y \Rightarrow T - 3x^2 = 0$$

$$\Rightarrow x^* = y^* = \sqrt{T/3} \Rightarrow z^* =$$

Extrema of function subject to one equality constraint

Q. Would previous approach work for *any* function *f* and constraint *g*?
A. No! Too "crude", sometimes can't express variable in terms of others.

Better approach: consider the geometry of the problem.

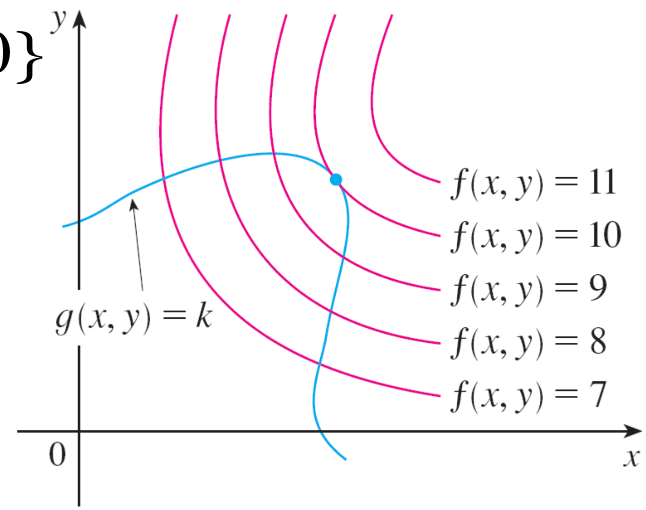
1) Suppose we have a function $f(\boldsymbol{x})$ of n-variables. The constraint $g(\boldsymbol{x}) = 0$ is a level surface in n-dimensional space.

2a) At any point on the constraint (level) surface, $\nabla g(\boldsymbol{x}) \perp \{g(x) = 0\}$

2b) For any curve **r(t)** on the level surface, at any point, $\nabla g(\boldsymbol{x}_0) \perp r'(t_0)$

3) If $f(\boldsymbol{x})$ has extreme value at point $\mathbf{x_0}$ on the constraint surface,

$$\nabla f(\boldsymbol{x}_0) \perp r'(t_0) \; or \; \nabla f(\boldsymbol{x}_0) \perp \{g(x_0) = 0\}$$



Otherwise can increase value of $f(\boldsymbol{x})$ by moving along the constraint surface.

4) There must exist a parameter λ:

$$\nabla f(\boldsymbol{x}_0) \parallel \nabla g(\boldsymbol{x}_0) \Rightarrow \lambda \neq 0$$

$$\nabla f(\boldsymbol{x}_0) + \lambda \nabla g(\boldsymbol{x}_0) = 0$$

Method of Lagrange Multipliers

A. Define a function: **L** is the Lagrangian, λ is the Lagrange multiplier
B. Find the critical points of **L** w.r.t {**x, λ**}
C. Obtain n+1 equations (n = dimension)
➔ n equations to determine extremum $\mathbf{x}^*$ (one equation for λ)
➔ might not care about λ, hence can be eliminated (undetermined multiplier)

$$A)\ \ L(\boldsymbol{x},\lambda) = f(\boldsymbol{x}) + \lambda g(\boldsymbol{x}),\ \ \lambda \neq 0$$

$$B)\ \ \nabla_x L = 0,\ \ \frac{\partial L}{\partial \lambda} = 0 \Rightarrow \nabla f(\boldsymbol{x}) + \lambda \nabla g(\boldsymbol{x}) = 0,\ \ g(\boldsymbol{x}) = 0$$

What is the maximum volume of a box?

Problem: find the max volume of a box without lid made from T(m²) amount of material.
Solution: use Method of Lagrange Multipliers.

$$\max_{\boldsymbol{x}} f(\boldsymbol{x}) \quad \text{s.t. } g(\boldsymbol{x}) = 0$$

$$f(\boldsymbol{x}) = V(x, y, z) = xyz$$

$$g(\boldsymbol{x}) = g(x, y, z) = 2xz + 2yz + xy - T$$

$$A)\ \ L(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) + \lambda g(\boldsymbol{x}),\ \ \lambda \neq 0$$

$$B)\ \ \nabla f(\boldsymbol{x}) + \lambda \nabla g(\boldsymbol{x}) = 0,\ \ g(\boldsymbol{x}) = 0$$

$$\nabla f(\boldsymbol{x}) = \langle yz, xz, xy \rangle \ \ \nabla g(\boldsymbol{x}) = \langle 2z + y, 2z + x, 2x + 2y \rangle$$

$$C)\ \ \langle yz, xz, xy \rangle = -\lambda \langle 2z + y, 2z + x, 2x + 2y \rangle$$

$$2xz + 2yz + xy - T = 0$$

Extrema of function subject to two equality constraints

Q. Can we generalize the Method of Lagrange Multipliers?

Suppose we seek the extrema of a function $f(\boldsymbol{x})$ of n-variables subject to **J** number of constraints $g_j(\boldsymbol{x}) = 0$ (each is a level surface in n-dimensions).

Geometrically, $\boldsymbol{x}$ is restricted to lie on the surface (curve) of intersection of the level surfaces $g_j(\boldsymbol{x})$.

At any point, the gradients of the constraint surfaces are orthogonal to the surface of intersection.

The gradient of $f(\boldsymbol{x})$ is in the space (plane) defined by the gradients of $g_j(\boldsymbol{x})$.

Formally stated, we have a more general problem:

$$\max_{\boldsymbol{x}} f(\boldsymbol{x}) \quad \text{s.t. } g_j(\boldsymbol{x}) = 0, \quad j = 1, \ldots, J$$

$$L(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) + \sum_{j=1}^{J} \lambda_j g_j(\boldsymbol{x}), \ \forall j, \lambda \neq 0$$

$$\nabla_x L = 0, \ \nabla_{\lambda_j} L = 0$$

$$\nabla f(\boldsymbol{x}) + \sum_{j=1}^{J} \lambda_j \nabla g_j(\boldsymbol{x}) = 0, \ g_j(\boldsymbol{x}) = 0$$