Math S1201
Calculus 3
Chapters 14.5 – 14.7

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Outline

- CH 14.5 The Chain Rule
  - Multivariate function where each variable is function of one variable
  - Multivariate function where each variable is function of two variables
  - Tree diagram
  - Implicit differentiation for multivariate functions
- CH 14.6 Directional Derivatives and Gradient
  - Directional derivative - definition
  - Gradient vector – definition
  - Generalization to n dimensions
  - Maximum rate of change
  - Significance of gradient

Outline

- CH 14.7 Maximum and Minimum Values
  - Local and global extrema
  - Extrema for functions of one variable
  - Critical points
  - Fermat's Theorem
  - Extrema for functions of two variables
  - 2^{nd} Derivatives test + proof
  - Determining global (absolute) extrema

Guiding Eyes (14.5)

A. How do you differentiate a multivariate function where each variable is a function of one variable?

B. How do you differentiate a multivariate function where each variable is a function of two variables?

C. How do you differentiate a multivariate function implicitly?

Chain rule for functions where each variable is a function of one variable

Consider functions of one variable: ***y = f(x)*** where ***x = g(t)***

1) Differentiate using chain rule.

What assumptions have we made? Functions are differentiable.

Proof of rule (section 3.4, p.204):

2) Apply property of increments to x = g(t), and y = f(x), where we assume that ***b = g(a)***.

$$1)\ \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$2)\ \Delta y = f'(a)\Delta x + \varepsilon \Delta x \quad where, \varepsilon \to 0 \quad as \quad \Delta x \to 0$$

$$2a)\ \Delta x = g'(a)\Delta t + \varepsilon_1 \Delta t \quad where, \varepsilon_1 \to 0 \quad as \quad \Delta t \to 0$$

$$2b)\ \Delta y = f'(b)\Delta x + \varepsilon \Delta x \quad where, \varepsilon_2 \to 0 \quad as \quad \Delta x \to 0$$

$$3)\ \Delta x = \left[g'(a) + \varepsilon_1\right]\Delta t$$

$$\Delta y = \left[f'(b) + \varepsilon_2\right]\Delta x = \left[f'(b) + \varepsilon_2\right]\left[g'(a) + \varepsilon_1\right]\Delta t$$

*Chain rule for functions where each variable is a function of **one** variable*

Consider functions of one variable: ***y = f(x)*** where ***x = g(t)***
2) Apply property of increments to x = g(t), and y = f(x), where we assume that ***b = g(a)***.
3) Simplify and substitute expressions into eq.   4) Take limit.

$$2a)\ \Delta x = g'(a)\Delta t + \varepsilon_1 \Delta t \quad where, \varepsilon_1 \to 0 \quad as \quad \Delta t \to 0$$

$$2b)\ \Delta y = f'(b)\Delta x + \varepsilon \Delta x \quad where, \varepsilon_2 \to 0 \quad as \quad \Delta x \to 0$$

$$3)\ \Delta x = \left[g'(a) + \varepsilon_1 \right] \Delta t$$

$$\Delta y = \left[f'(b) + \varepsilon_2 \right] \Delta x = \left[f'(b) + \varepsilon_2 \right]\left[g'(a) + \varepsilon_1 \right] \Delta t$$

$$4)\ \lim_{\Delta t \to 0} \Rightarrow \Delta x \to 0 \Rightarrow \varepsilon_1, \varepsilon_2 \to 0$$

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} \left[f'(b) + \varepsilon_2 \right]\left[g'(a) + \varepsilon_1 \right] = f'(b) g'(a)$$

$$= f'\big(g(a)\big) g'(a)$$

*Chain rule for **multivariate** functions where each variable is a function of **one** variable*

Consider functions of two variables: ***z = f(x,y)*** where ***x = g(t), y = h(t)***
Can we differentiate using chain rule?
What assumptions have we made? Functions (f,g,h) are differentiable.
When is ***f(x,y)*** differentiable? Partial derivatives are cont.
Proof of rule:
1) Since ***f*** is differentiable, can use property for dz.
2) Since ***g,h*** are differentiable, when Δt goes to zero, so do Δx, Δy.
3) Substitute by definition.

$$1)\ \Delta z = \frac{\partial f}{\partial x}\Delta x + \varepsilon_1 \Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_2 \Delta y$$

$$where, \varepsilon_1, \varepsilon_2 \rightarrow 0 \ \ as \quad \Delta x, \Delta y \rightarrow 0$$

$$\Delta z = \left[\frac{\partial f}{\partial x} + \varepsilon_1\right]\Delta x + \left[\frac{\partial f}{\partial y} + \varepsilon_2\right]\Delta y$$

Consider functions of two variables: ***z = f(x,y)*** where ***x = g(t), y = h(t)***

1) Since ***f*** is differentiable, can use property for dz.
2) Since ***g,h*** are differentiable, when Δt goes to zero, so do Δx, Δy.
3) Substitute by definition.

$$1)\ \Delta z = \frac{\partial f}{\partial x}\Delta x + \varepsilon_1 \Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_2 \Delta y$$

$$where, \varepsilon_1, \varepsilon_2 \to 0 \quad as \quad \Delta x, \Delta y \to 0$$

$$\Delta z = \left[\frac{\partial f}{\partial x} + \varepsilon_1\right]\Delta x + \left[\frac{\partial f}{\partial y} + \varepsilon_2\right]\Delta y$$

$$2)\ \frac{dz}{dt} = \lim_{\Delta t \to 0}\frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0}\left[\frac{\partial f}{\partial x} + \varepsilon_1\right]\frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0}\left[\frac{\partial f}{\partial y} + \varepsilon_2\right]\frac{\Delta y}{\Delta t}$$

$$\Delta t \to 0 \Rightarrow \Delta x, \Delta y \to 0 \qquad \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} = \frac{dx}{dt}, \ \lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t} = \frac{dy}{dt}$$

$$3)\ \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

*Chain rule for **multivariate** functions where each variable is a function of **two** variables*

Consider functions of two variables: ***z = f(x,y)*** where ***x = g(s,t), y = h(s,t)***
Can we differentiate using chain rule?
We would like to compute partial derivatives of **z** w.r.t **s** & **t**.
Treat each partial separately, and apply previously derived rule.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \Rightarrow \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Q. Does the rule generalize to n variables?
A. As long as all functions are differentiable.

Concept: a tree diagram describes the relation between dependent / intermediate / independent variables.

How do you differentiate multivariate functions implicitly?

Consider functions of one variable: sometimes it is difficult to express **y** as a function of **x** explicitly. We can define function(s) implicitly instead (ch 3.5).
Simple example: circle.
Suppose ***F(x,y) = 0*** defines ***y*** implicitly as a differentiable function of **x**.

$$\forall x \in D_f,\ \ F\left(x, f(x)\right) = 0$$

Q. When is this assumption valid?
A. Implicit Function Theorem (IFT): exclude pathological cases.
F is defined on a disk containing point (a,b), $F\left(a,b\right) = 0,\ \ F_y(a,b) \neq 0$
and partial derivatives are cont. on the disk, then assumption is valid near the point (a,b).

$$z = F\left(x,y\right) = 0,\ x = g(x),\ y = f(x)$$

If assumption holds?
Case 1 of chain rule.

$$\frac{dz}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} \ \Rightarrow 0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$$\frac{\partial F}{\partial y} \neq 0 \ \Rightarrow\ \frac{dy}{dx} = -\frac{\partial F}{\partial x} / \frac{\partial F}{\partial y} = -\frac{F_x}{F_y}$$

How do you differentiate multivariate functions implicitly?

What is the general approach? 1) Differentiate w.r.t **x** 2) Solve eq. for ***f'(x)***.
Consider functions of two variables: we can define **z** implicitly as a differentiable function of ***f(x,y)***: ***F(x,y, f(x,y)) = 0***.

Q. When is this assumption valid?

A. IFT: instead of a disk, we need a sphere containing point (a,b,c). Partial derivatives are cont. on the disk, then assumption is valid near the point.

$$F\left(a,b,c\right)=0,\ \ F_z(a,b,c)\neq 0$$

If assumption holds? use chain rule twice (for x and y).

$$u=F\left(x,y,z\right)=0,\, x=g(x),\ y=f(y),z=f(x,y)$$

$$\frac{du}{dx}=\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\ \Rightarrow 0=\frac{\partial F}{\partial x}+0+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

$$\frac{\partial F}{\partial z}\neq 0\ \Rightarrow\ \frac{\partial z}{\partial x}=-\frac{\partial F}{\partial x}/\frac{\partial F}{\partial z}=-\frac{F_x}{F_z}\quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

Guiding Eyes (14.6)

A. How do you compute the directional derivative?

B. In which direction does the function change fastest, and what is the maximum rate of change?

C. What is the significance of the gradient vector?

D. What is the equation of the tangent plane to the level surface?

How do you compute the directional derivative?

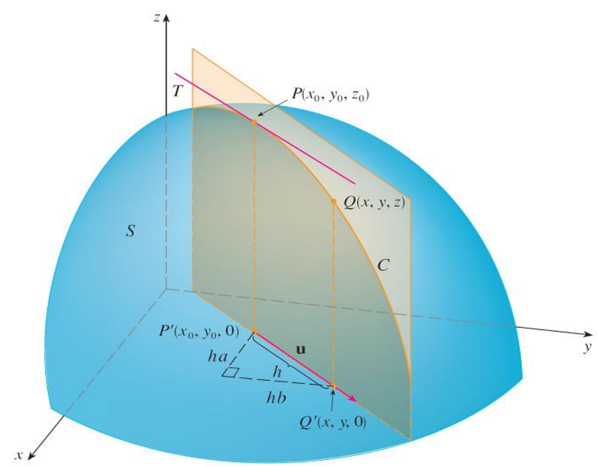
Recall from section 14.3:

1) Choose a point (a,b,c) on the surface S, ***z = f(x,y)***.
2) Choose any arbitrary direction **u** (unit vector).
3) The direction we consider is a line which determines a plane in space. We restrict our attention to the trace of the surface S in the plane.
4) The directional derivative is the slope of the tangent line to the trace. The "rate of change of z in the direction of **u**".

$$D_u f(x,y) = \lim_{h\to 0} \frac{f(x+hu_1, y+hu_2) - f(x,y)}{h}$$

$$\boldsymbol{u} = \langle 1,0 \rangle \Rightarrow D_u f = D_1 f = f_x = \frac{\partial f}{\partial x}$$

$$\boldsymbol{u} = \langle 0,1 \rangle \Rightarrow D_u f = D_2 f = f_y = \frac{\partial f}{\partial y}$$



How do you compute the directional derivative?

We need a simple formula to compute the directional derivative for any **u**. We use a familiar trick to define a function g(h), compute g'(h) using the chain rule, and show that g'(0) is by definition the directional derivative.

$$g(h) = f(x + hu_1, y + hu_2) \Rightarrow g(0) = f(x, y)$$

$$D_u f(x, y) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0)$$

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{\partial h} + \frac{\partial f}{\partial y}\frac{dy}{dh} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2$$

$$g'(0) = f_x(x, y)u_1 + f_y(x, y)u_2$$

$$D_u f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle u_1, u_2 \rangle$$

$$D_u f(x, y) = \nabla f(x, y) \cdot \boldsymbol{u}$$

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

How do you compute the directional derivative?

Problem: find the directional derivative at point P where **u** makes an angle θ with the positive x-axis.

Solution: by definition, $\boldsymbol{u} = \langle \cos\theta, \sin\theta \rangle$

Compute using the formula: $D_u f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$

Concept: the gradient vector of a function is the vector of 1st partial derivatives.

Problem: find directional derivative at point P in the direction of vector **v**.

Solution: 1) Compute gradient vector (partial derivatives) at point P.

2) Convert **v** to a unit vector (divide by magnitude)

3) Compute the dot product of the two vectors.

Q. Does the definition apply to functions of 3 (n) variables?

A. Add dimensions (components) to your vectors.

In which direction does the function change fastest?

By definition of dot product, the maximum value of the directional derivative occurs when **u** is in the same direction as the gradient. Therefore, the gradient is the direction of maximal increase, and the maximal rate of change is the magnitude of the gradient.

$$|\boldsymbol{u}| = 1 \Rightarrow D_u f = \nabla f \cdot \boldsymbol{u} = |\nabla f||\boldsymbol{u}|\cos\theta = |\nabla f|\cos\theta$$

$$\cos\theta = 1 \;\Rightarrow\; \theta = 0$$

Problem: given ***z = f(x,y)*** find the rate of change at point P in the direction of point Q. In what direction is the rate maximal? What is its value?

Solution: 1) Compute gradient vector (partial derivatives) at point P.

2) Convert a unit vector **u** (divide by magnitude) in the direction of PQ.

3) Compute the dot product of the two vectors.

4) Maximal rate of change is in direction of grad. Value is |g| at P.

Equation of tangent plane to level surface

Consider the level surface S, ***F(x,y,z) = k***.
Let r(t) denote any curve on S that passes through point P(x_0,y_0,z_0).
We assume all functions are differentiable, and use the chain rule.
We observe that the gradient is perpendicular to the tangent vector r'(t).
We can define the tangent plane to the level surface at P (grad = normal).

$$F(x,y,z)=k \qquad r(t)=\langle x(t),y(t),z(t)\rangle$$

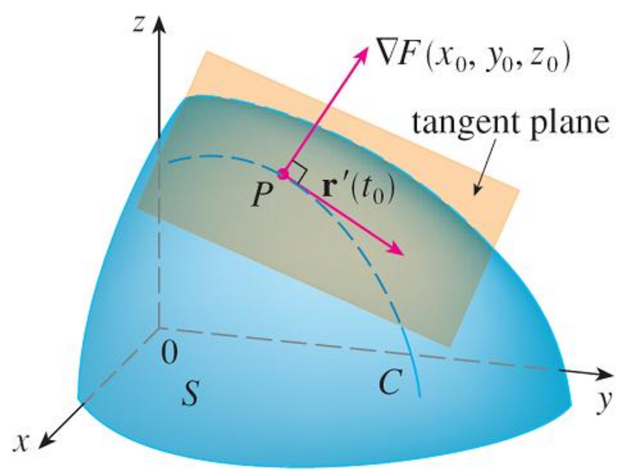
$$\nabla F=\langle F_x,F_y,F_z\rangle \qquad r'(t)=\langle x'(t),y'(t),z'(t)\rangle$$

$$\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}=0$$

$$\nabla F\cdot r'(t)=0$$

$$\nabla F(x_0,y_0,z_0)\cdot\langle x,y,z\rangle-\langle x_0,y_0,z_0\rangle$$

$$F(x,y,z)=f(x,y)-z=0\Rightarrow F_z=-1$$

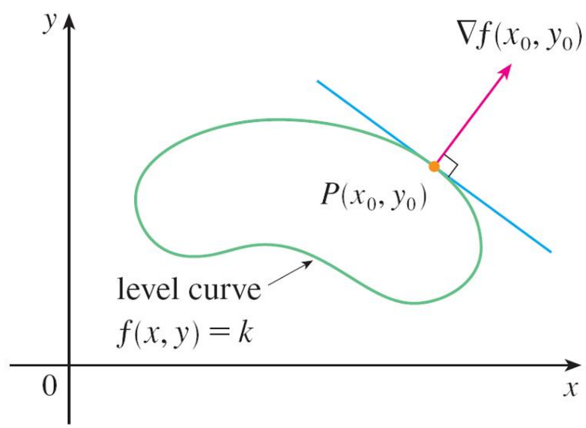


What is the significance of the gradient vector?

Consider a function *f* of two variables:

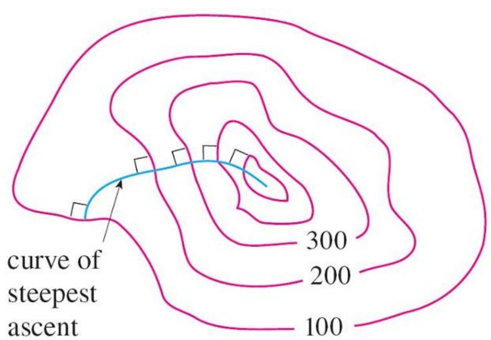
1) The gradient is the direction of maximal increase.
2) The gradient is perpendicular to the level curve *f(x,y) = k*.

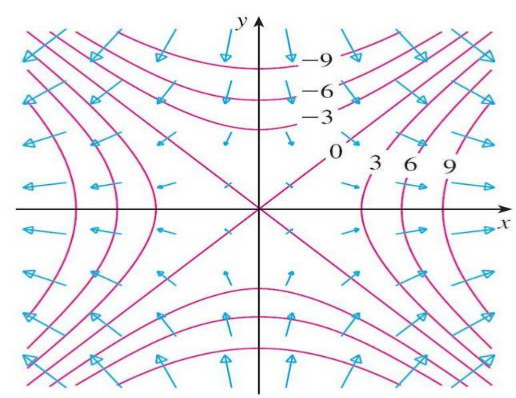
Intuition: as we move away from a point along the level curve, the value of *f* remains constant. If we move in the perpendicular direction, we expect to get the maximum increase.



A curve of steepest ascent can be drawn by making it perpendicular to all the contour lines. A plot of each grad vector for a set of points is called a *gradient vector field*.

Example: gradient vector field for the function superimposed on a contour map of *f*.





Guiding Eyes (14.7)

A. How do you identify the extremum of a function?

B. How does the 2nd derivative test generalize to functions of two variables?

C. How do you determine whether the extremum is absolute (global)?

How do you identify the extremum of a function?

Concept: a function *f* has a local extremum at a point P if its value at P is larger / smaller than *nearby* values of *f.*

Concept: a function *f* has a global extremum at a point P if its value at P is larger / smaller than *all* values of *f.*

Consider functions of one variable: ***y = f(x)*** at point ***x = a***:

If ***f'(a) = 0*** or ***f'(a)*** *d.n.e* then we MIGHT have extremum at **a**.

If extremum exists at **a**, then we MUST have ***f'(a) = 0*** or ***f'(a)*** *d.n.e*.

Concept: a function *f* has a critical (stationary) point P if the 1st derivative(s) at P are zero or undefined.

Fermat's Theorem (Section 4.1):

Assuming that,

1) *f* has a local extremum at **a.**
2) *f'(a)* exists.

It follows that ***f'(a) = 0***

How do you identify the extremum of a function?

Consider functions of one variable: ***y = f(x)*** at point ***x = a***:
If ***f'(a) = 0*** or ***f'(a)*** *d.n.e* then we MIGHT have extremum at **a**.
If extremum exists at **a**, then we MUST have ***f'(a) = 0*** or ***f'(a)*** *d.n.e*.

Fermat's Theorem (Section 4.1):
Assuming *f* has a local extremum at **a** and *f'(a)* exists, implies ***f'(a) = 0***.
Proof: we "squeeze" f'(a) to zero.
Consider the LH and RH limits. These limits must equal the two-sided limit due to assumption (2). Approaching from one side we bound f'(a) from above, approaching from the other side, we bound f'(a) from below.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

$$h > 0 \Rightarrow f'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \leq \lim_{h\to 0^+} 0 = 0$$

$$h < 0 \Rightarrow f'(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \geq \lim_{h\to 0^-} 0 = 0$$

How do you identify the extremum of a function?

Q. If we have a critical point, how can we confirm that it is indeed an extremum?

A. 2nd derivative test:

If ***f''(a) > 0*** or ***f''(a) < 0*** then we have an extremum at **a**.

If ***f''(a) = 0*** or ***f''(a) d.n.e*** then test is inconclusive.

Q. What should you do if 2nd test is inconclusive?

A. 1st derivative test: does f'(x) change sign around **a**.

Consider functions of two variables: ***z = f(x,y)*** at point ***x = (a,b)***:

> **1 Definition** A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

Geometric interpretation: tangent plane at extremum is horizontal.

How do you identify the extremum of a function?

Example:

$f(x, y) = x^2 + y^2$ Normal to tangent plane is in direction of z-axis.

$f_x(0, 0) = 0$ and $f_y(0, 0) = 0$ → minimum at (0,0).

Example:

$f(x, y) = y^2 - x^2$ → $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$.

But (0, 0) is not an extreme value (its a saddle point)!

Consider functions of two variables: ***z = f(x,y)*** at point ***x = (a,b)***:

If $\boldsymbol{f_x(a,b) = f_y(a,b) = 0}$ or at least one of the partials *d.n.e* then we MIGHT have extremum at **(a,b)**.

> **2** **Fermat's Theorem for Functions of Two Variables** If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

How do you identify the extremum of a function?

Consider functions of two variables: $\boldsymbol{z = f(x,y)}$ at point $\boldsymbol{x = (a,b)}$:
If $\boldsymbol{f_x(a,b) = f_y(a,b) = 0}$ or at least one of the partials *d.n.e* then we MIGHT have extremum at **(a,b)**.

> **2 Fermat's Theorem for Functions of Two Variables** If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Recall Fermat's Theorem (Section 4.1):
Assuming *f* has a local extremum at **a** and *f'(a)* exists, implies $\boldsymbol{f'(a) = 0}$.

Proof of extended version:
1) Fix **y = b**, and consider the partial derivative w.r.t **x**. By assumption, $\boldsymbol{g(x) = f(x,b)}$ has extremum at **(a,b)** and $\boldsymbol{f_x(x,b) = g'(x)}$ exists.
2) Apply Fermat's TH to g(x). Conclude that $\boldsymbol{g'(a) = f_x(a,b) = 0}$.
3) Repeat argument but fix **x = a**, and consider the partial derivative w.r.t **y**.

2nd Derivatives Test

Q. Given critical point of a multivariate function, is it an extremum?

A. Non-trivial generalization of the 2nd derivative test.

For functions of one variable:

If ***f''(a) > 0*** or ***f''(a) < 0*** then we have an extremum at **a**.

If ***f''(a) = 0*** or ***f''(a) d.n.e*** then test is inconclusive.

Now we have more than one 2nd derivative (partials)!

Idea: evaluate the 2nd directional derivative, and show that it is positive (negative) for any choice of direction.

$$D_u f = f_x u_1 + f_y u_2$$

$$D_u^2 f = D_u(D_u f) = \frac{\partial}{\partial x}(D_u f)\,u_1 + \frac{\partial}{\partial y}(D_u f)u_2$$

$$= (f_{xx}u_1 + f_{yx}u_2)\,u_1 + (f_{xy}u_1 + f_{yy}u_2)u_2$$

$$= f_{xx}u_1^2 + f_{yx}u_2u_1 + f_{xy}u_1u_2 + f_{yy}u_2^2$$

$$= f_{xx}u_1^2 + 2f_{xy}u_1u_2 + f_{yy}u_2^2$$

2nd Derivatives Test

Idea: evaluate the 2nd directional derivative, and show that it is positive (negative) for any choice of direction.
Use Clairaut's TH. to combine mixed partial derivatives, and complete square.

$$D_u f = f_x u_1 + f_y u_2$$

$$D_u^2 f = f_{xx} u_1^2 + 2 f_{xy} u_1 u_2 + f_{yy} u_2^2$$

$$= f_{xx}\left(u_1^2 + 2\frac{f_{xy}}{f_{xx}} u_1 u_2 + \frac{f_{yy}}{f_{xx}} u_2^2 \right)$$

$$= f_{xx}\left(u_1 + \frac{f_{xy}}{f_{xx}} u_2 \right)^2 + \frac{u_2^2}{f_{xx}}\left(f_{xx} f_{yy} - f_{xy}^2 \right)$$

$$f_{xx} > 0, D = \left(f_{xx} f_{yy} - f_{xy}^2 \right) > 0 \Rightarrow D_u^2 f > 0$$

$$f_{xx} < 0, D = \left(f_{xx} f_{yy} - f_{xy}^2 \right) > 0 \Rightarrow D_u^2 f < 0$$

2^{nd} Derivatives Test

3 **Second Derivatives Test** Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Note: case (c) is the "saddle point".

Note: if D = 0, the test gives no information.

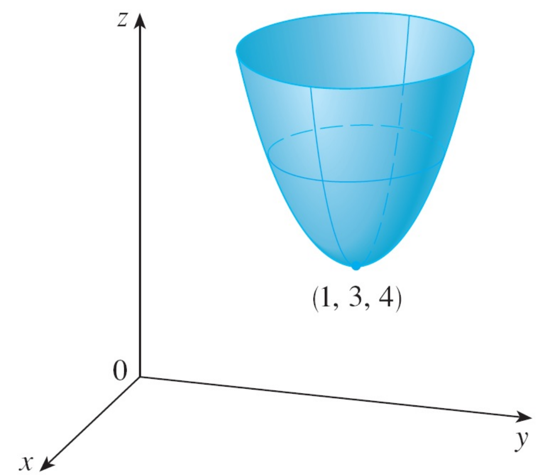
Concept: the matrix of 2^{nd} partial derivatives is called the Hessian.

$$D = \left(f_{xx} f_{yy} - f_{xy}^2\right) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

How do you identify the extremum of a function?

Problem: find local extrema / saddle points of $z = f(x,y)$.
Solution: 1) Compute partial derivatives.
2) Solve the resulting equations simultaneously to locate critical points.
3) Calculate the 2nd partial derivatives and D.
4) Determine which case of 2nd derivatives test applies.

Example:
$f(x, y) = x^2 + y^2 - 2x - 6y + 14$.
1) $f_x(x, y) = 2x - 2$ $f_y(x, y) = 2y - 6$
2) Critical point is (1, 3)
Completing the square,
$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$
Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have
$f(x, y) \geq 4$ for all values of x and y.
Therefore $f(1, 3) = 4$ is a local minimum



How do you identify the extremum of a function?

Problem: find local extrema / saddle points of $z = f(x,y)$.
Solution: 1) Compute partial derivatives.
2) Solve the resulting equations simultaneously to locate critical points.
3) Calculate the 2nd partial derivatives and D.
4) Determine which case of 2nd derivatives test applies.

Example:
$f(x, y) = x^4 + y^4 - 4xy + 1$.
1) $f_x(x, y) = 4x^3 - 4y$ $\quad f_y(x, y) = 4y^3 - 4x$
2) $x^3 - y = 0 \quad y^3 - x = 0$ $\quad$ Critical points are (0, 0) (1,1) (-1,-1)
3) $f_{xx}(x, y) = 12x^2$ $\quad f_{xy}(x, y) = 4$ $\quad f_{yy}(x, y) = 12y^2$ $\quad D = 144x^2y^2 - 16$
4)
$D(0,0) = -16 < 0$ $\quad$ (0,0)→ saddle point
$D(1,1) = 128 > 0$ $\quad f_{xx}(1,1) = 12 > 0$ $\quad$ (1,1)→ local minimum
$D(-1,-1) = 128 > 0$ $\quad f_{xx}(-1,-1) = 12 > 0$ $\quad$ (1,1)→ local minimum

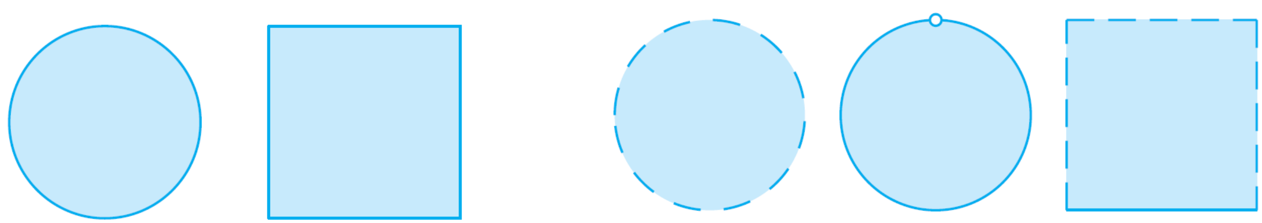
How do you determine a global extremum?

Consider functions of one variable: ***y = f(x)***:
Extreme Value Theorem: if *f* is cont. on the closed interval [a,b], then *f* has an absolute (global) minimum and maximum values on the interval.
To determine global extrema, evaluate *f* at critical points and interval end points.

Consider functions of two variables: ***z = f(x,y)***:
Consider closed set (contains all boundary points) instead of interval.

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To determine global extrema, evaluate *f* at critical points and boundary curves.



Closed sets

Sets that are not closed

How do you determine a global extremum?

Problem: find global extrema of ***z = f(x,y)*** on the domain D.

Solution: 1) Verify that *f* is cont. on a closed bounded set.

2) Compute partial derivatives.

3) Solve the resulting equations simultaneously to locate critical points.

4) Evaluate function values at critical points, and on the boundary.

5) Find largest / smallest of the values in step (4).

Example:

$f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \le x \le 3, 0 \le y \le 2\}$.

2) $f_x = 2x - 2y = 0$ $\quad$ $f_y = -2x + 2 = 0$

3) x = 1 Critical point is (1,1) $f(1,1) = 1$

4) Boundary: $f(x,0) = x^2$ $\quad f(x,2) = x^2 - 4x + 4$ $\quad 0 \le x \le 3$

$f(0,y) = 2y$ $\quad f(3,y) = 9 - 4y$ $\quad 0 \le y \le 2$

minimum: $f(x,0) = x^2 = 0$ $\quad f(x,2) = 0$ $\quad f(0,y) = 0$ $\quad f(3,y) = 1$

maximum: $f(x,0) = x^2 = 9$ $\quad f(x,2) = 4$ $\quad f(0,y) = 4$ $\quad f(3,y) = 9$

5) Global maximum: f(3,0) = 9 $\quad$ Global minimum: f(0,0) = f(2,2) = 0