Math S1201
Calculus 3
Chapters 14.2 – 14.4

Summer 2015
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Outline

- CH 14.2 Multivariate Functions: Limits and Continuity
  - Limits along paths
  - Determining if a limit exists
  - Continuous functions
  - Functions of n variables
- CH 14.3 Partial Derivatives
  - Interpretation of partial derivatives
  - Derivative in simplest (x,y) direction
  - Rules for finding partial derivatives
  - 2^{nd} partial derivatives
  - Mixed partial derivative equality
  - Clairut's Theorem: proof
  - Higher order partial derivatives

Outline

- CH 14.4 Linear Approximations & Tangent Planes
  - Tangent plane
  - Linear approximations (linearization)
  - Differentials
  - Functions that behave badly
  - Differentiable functions

Guiding Eyes (14.2)

A. **Does the concept (& definition) of limit generalize to functions of two variables?**

B. **Does the concept (& definition) of continuity generalize to functions of two variables?**

C. **Do the concepts of limit and continuity generalize to functions of n-variables?**

Does limit concept & definition generalize to functions of two variables?

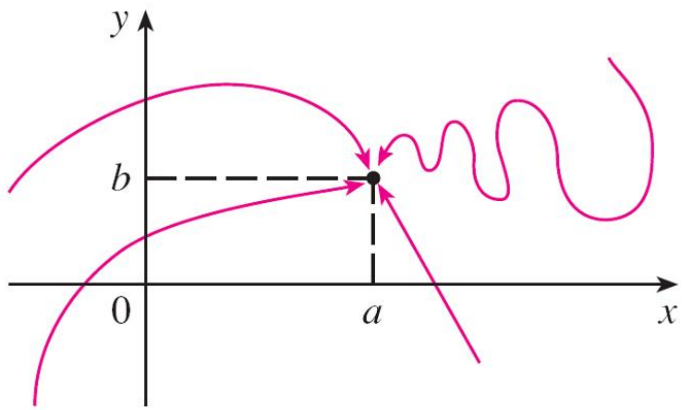
1) Modify notation to account for 2nd variable.
2) As for a single variable, limit may not exist.
3a) Single variable: ***x*** can approach ***a*** from TWO directions (right/left +/-).
3b) Two variables: (***x,y)*** can approach ***(a,b)*** from infinitely many directions.
4a) Single variable: if limit exists, *f*(***x***) must approach ***L*** from both directions.
4b) Two variables: if limit exists, *f*(***x,y***) must approach ***L*** from any path.
5) "Corollary": if we can find (any) two paths with different limits along them, the limit does not exists at the point.

Q. How does the definition generalize?

A. Replace interval with distance, introduce disk with center at point of interest.

$$\lim_{x\to a} f(x) = L \;\Rightarrow\; \lim_{(x,y)\to(a,b)} f(x,y) = L$$

$$|x-a| \;\Rightarrow \sqrt{(x-a)^2 + (y-b)^2}$$



1 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

5 If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

Related Problems

Problem: given function of two variables, show that the limit does not exist (d.n.e) at some point.
Solution: find two (simple) paths (x/y-axis) along which the limit is different.

Example:

$f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along the x-axis
$f(x, y) \to -1$ as $(x, y) \to (0, 0)$ along the y-axis

$$\lim_{(x, y)\to(0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$$

Problem: given function of two variables, does limit exist at some point?
Solution:

1) If you can find a counter-example (previous problem), limit d.n.e
2) Considering paths along any straight line is not sufficient!

Example:

$f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along $y = mx$
$f(x, y) \to 1/2$ as $(x, y) \to (0, 0)$ along $x = y^2$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$$

Related Problems

Problem: given function of two variables, does limit exist at some point?
Solution: (continued)
3) If you suspect that limits exists, then use properties of limits (which can be extended) to to prove that conditions in (epsilon-delta) definition hold.
3a) Find limit along some path.
3b) Substitute that limit for L in definition, find delta for every epsilon (express in terms of if necessary). Conclude that limits exists.

Example: $$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} \qquad a=0, b=0, L=0$$

$$0<\sqrt{x^2+y^2}<\delta \to \left|\frac{3x^2y}{x^2+y^2}-0\right|<\varepsilon$$

$$\left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} = 3\sqrt{y^2}\,\frac{x^2}{x^2+y^2} \le 3\sqrt{y^2} \le 3\sqrt{x^2+y^2}$$

$$3\sqrt{x^2+y^2} < 3\delta \quad \Rightarrow \quad \delta = \frac{\varepsilon}{3}$$

Does continuity concept & definition generalize to functions of two variables?

Direct substitution property can be extended.

$$\lim_{x\to a} f(x) = f(a) \;\Rightarrow\; \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Function is cont. on the domain D, if cont. at every point (a,b) in D.
Combinations of cont. functions are cont.

Examples:

1) Polynomials: built up from simpler functions by + and *.
2) Rational functions: ratio of polynomials, quotient of cont. functions.
3) Composite functions: $h(x,y) = g(f(x,y))$ cont. if f and g (defined on range of f) are

Problem: given polynomial, evaluate limit.
Solution: since polynomial is cont. everywhere, find limit by direct subst.

Problem: given rational functions, where is it cont.?
Solution: cont. on its domain.

Guiding Eyes (14.3)

A. **How does the concept of derivative generalize to functions of two variables?**

B. **How does the concept of 2nd derivative generalize to functions of two variables?**

C. **Under what conditions are the mixed partial derivatives equal?**

How does concept of derivative generalize to functions of two variables?

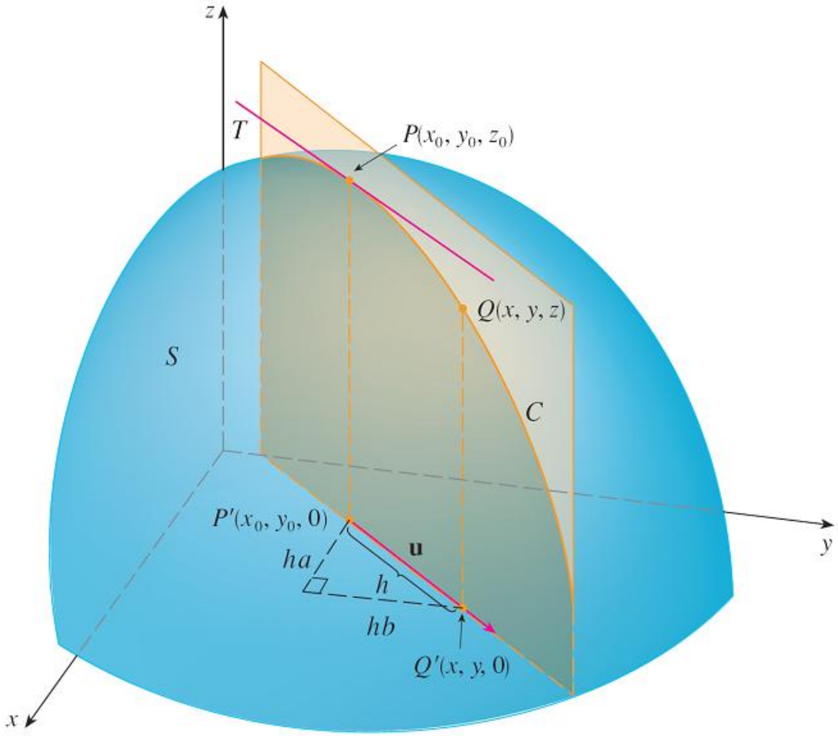
1) Choose a point (a,b,c) on the surface S, *z = f(x,y)*.
2a) Single variable: consider rate of change in one direction.
2b) Two variables: can choose any arbitrary direction (unit vector).

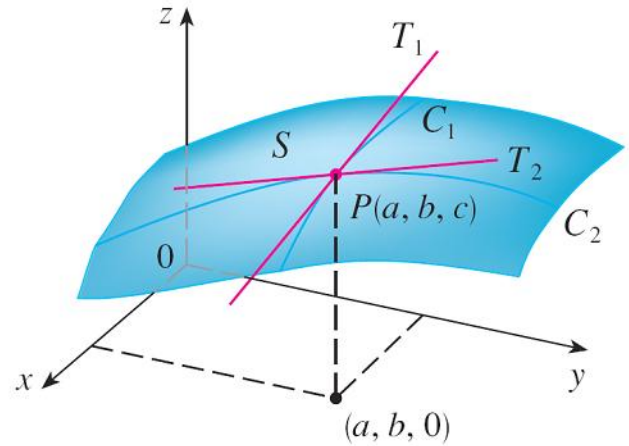
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$D_u f(x,y) = \lim_{h\to 0} \frac{f(x+hu_1, y+hu_2) - f(x,y)}{h}$$

3) The direction we consider is a line which determines a plane in space. We restrict our attention to the trace of the surface S in the plane. The directional derivative is the slope of the tangent line to the trace.
4) The two simplest directions: fix y/x variables (x = a or y = b). The traces are the intersections of the planes above with S. These are called partial derivatives with respect to x (y).

$$D_1 f = f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x,y)}{h}$$





Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

How does concept of derivative generalize to functions of two variables?

Problem: find the partial derivatives of a multivariate function.
Solution: keep other variable(s) constant, and differentiate a function of one variable.

$$f(x,y) = 4 - x^2 - 2y^2$$

$$f_x(x,y) = -2x \quad f_y(x,y) = -4y$$

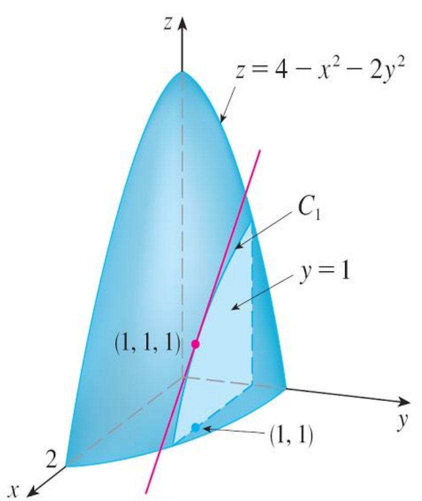
Problem: find the partial derivatives of a an implicitly defined function.
Solution: keep other variable(s) constant, and differentiate z as well. Then solve for partial of z w.r.t variable.

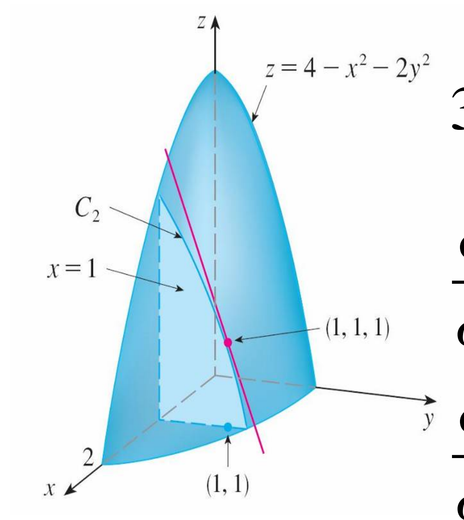
$$x^3 + y^3 + z^3 + 6xyz = 1$$

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(3z^2 + 6xy\right) = -3x^2$$

$$\frac{\partial z}{\partial x} = \frac{-x^2}{z^2 + 2xy}$$





How does concept of 2nd derivative generalize to functions of two variables?

More general question: higher order derivatives.
Since 1st partial derivative is also a function of 2 variables, same rules apply.

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

Problem: find 2nd partial derivatives.
Solution: hold appropriate variable constant, differentiate to obtain 1st partials, then repeat (4 total).

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

$$f_x = 3x^2 + 2xy^3 \qquad f_y = 3x^2y^2 - 4y$$

$$f_{xy} = 6xy^2 \qquad f_{yx} = 6xy^2$$

Q. Mixed partials derivatives above are equal, is this always true?
A. Clairaut's Theorem.

Under what circumstances are the mixed partial derivatives equal?

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Proof idea:

a) Show that $f_{xy}(a,b)$ is equal to some quantity H.

b) Show that $f_{yx}(a,b)$ is equal to the same quantity H.

c) Conclude that $f_{xy}(a,b) = f_{yx}(a,b)$

$$\begin{bmatrix} (a,b+h) & (a+h,b+h) \\ (a,b) & (a+h,b) \end{bmatrix}$$

Proof sketch:

A) Consider rectangle of coordinates.

B) Consider the segments:

$$(a+h,b+h) \to (a+h,b)$$

$$(a,b+h) \to (a,b)$$

By fixing first x, then y, we construct differences between function values.

$$g(x) = f(x,b+h) - f(x,b)$$

$$\Delta h = g(a+h) - g(a)$$

C) By applying MVT twice and taking the limit, we obtain the mixed partial derivative.

$$\begin{bmatrix} (a,b+h) & (a+h,b+h) \\ (a,b) & (a+h,b) \end{bmatrix}$$

$$(a+h,b+h) \rightarrow (a+h,b)$$

$$(a,b+h) \rightarrow (a,b)$$

$$g(x) = f(x,b+h) - f(x,b)$$

$$\Delta h = \left[f(1,2) - f(2,2) \right] - \left[f(1,1) - f(2,1) \right]$$

$$= g(a+h) - g(a)$$

$$\exists c : \frac{\Delta h}{h} = g'(c) = f_x(c,b+h) - f_x(c,b)$$

$$\exists d : \frac{\Delta h}{h^2} = \frac{f_x(c,b+h) - f_x(c,b)}{h} = f_{xy}(c,d)$$

$$h \rightarrow 0 \quad \Rightarrow (c,d) \rightarrow (a,b) \Rightarrow \lim_{h \rightarrow 0} \frac{\Delta h}{h^2} = f_{xy}(a,b)$$

$$f_{xy}(a,b) = \frac{\partial}{\partial y}\left(f_x(a,b)\right) = \lim_{h\to 0} \frac{f_x(a,b+h) - f_x(a,b)}{h}$$

$$f_x(a,b+h) = \lim_{h_1\to 0} \frac{f(a+h_1, b+h) - f(a,b+h)}{h_1}$$

$$f_x(a,b) = \lim_{h_2\to 0} \frac{f(a+h_2, b) - f(a,b)}{h_2}$$

$$f_{xy}(a,b) =$$

$$= \lim_{h\to 0} \frac{\displaystyle \lim_{h_1\to 0} \frac{f(a+h_1, b+h) - f(a,b+h)}{h_1} - \lim_{h_2\to 0} \frac{f(a+h_2, b) - f(a,b)}{h_2}}{h}$$

Guiding Eyes (14.4)

A. **Can you approximate a function of two variables with a linear function?**

B. **What happens if a function behaves "badly"?**

C. **How is the differential defined for a function of two variables?**

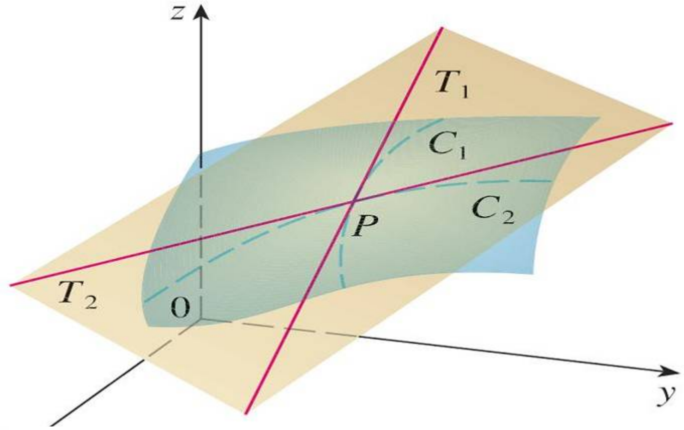
Can you approximate a function of two variables with a linear function?

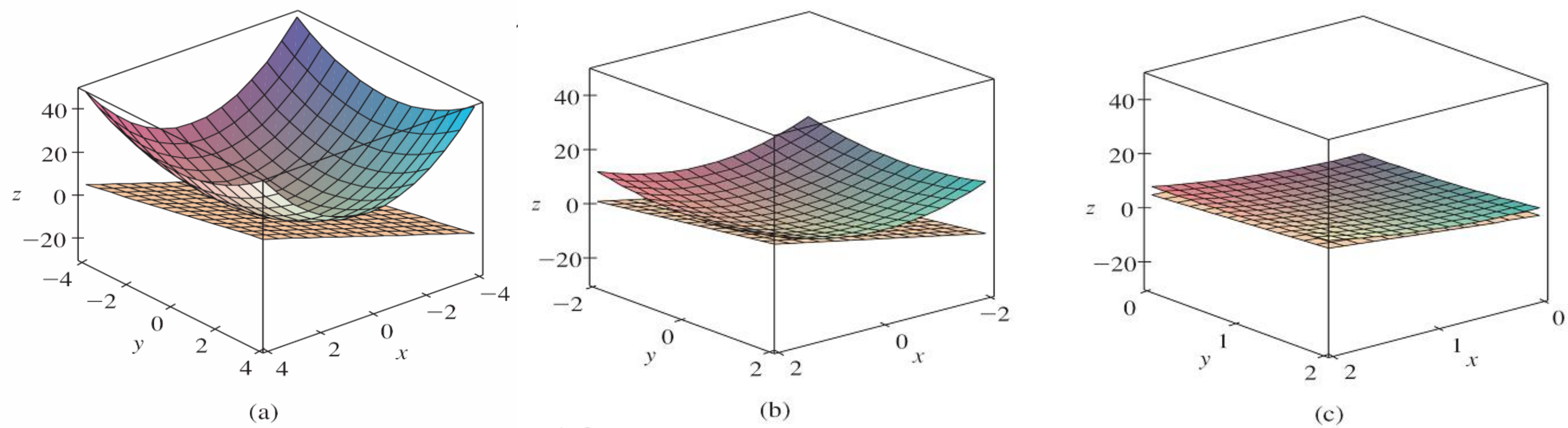
Idea: if we zoom in to a point on the graph of a differentiable function, the graph resembles the tangent line and can be approximated by a linear function. Equation of tangent line: $y - y_0 = f'(x_0)(x - x_0)$

Q. Does the idea generalize to functions of two variables?

1) Assume we have a surface S, $z = f(x,y)$, and 1st partial derivatives are cont. at point $P(x_0,y_0,z_0)$.
2) Obtain two curves by intersecting planes ($y = y_0$, $x = x_0$) with S.
3) We know the tangent lines to these curves.
4) The tangent plane is defined as the plane through point P that contains both tangent lines.
5) The tangent plane is the linear approximation to the function at P.

Q. What about any other intersecting curve?
A. Tangent plane contains all tangent lines.


z
T_1
C_1
C_2
P
T_2
0
y


40
20
z 0
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−4
−2
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y
2
4
4
2
0
x
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(a)
40
20
z 0
−20
−2
0
y
2
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20
z 0
−20
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1
x
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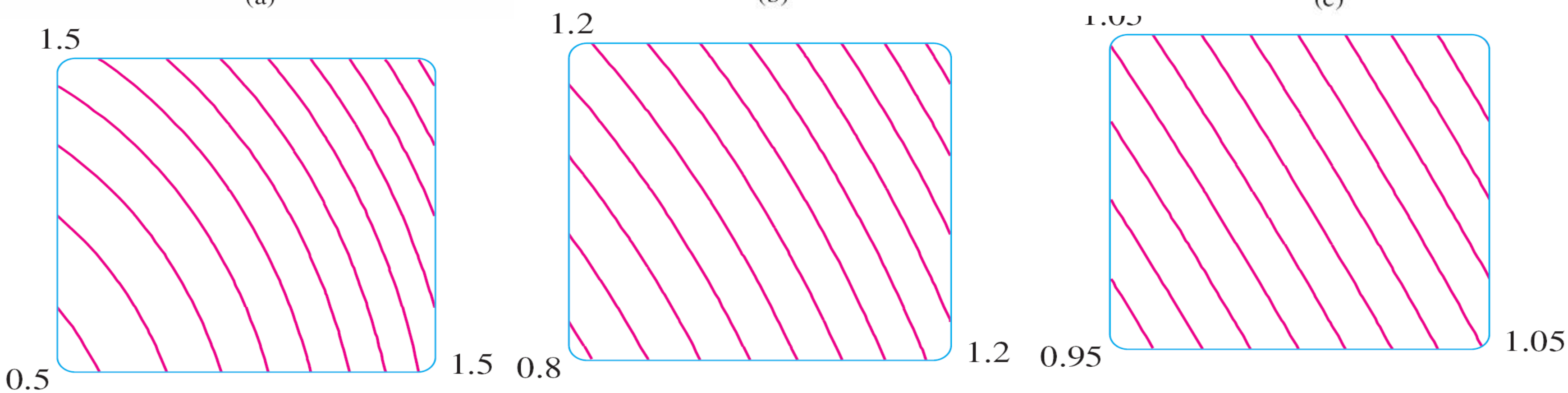

1.5
0.5
1.5
1.2
0.8
1.2
0.95
1.05

Can you approximate a function of two variables with a linear function?

Q. What is the equation of the tangent plane at point P_0?

1a) Eq. of tangent line to C1 is the eq. of line in x = x_0 plane.
1b) Eq. of tangent line to C2 is the eq. of line in y = y_0 plane.
2a) Obtain tangent vector to to C1:
2b) Obtain tangent vector to to C2:
3) Compute normal vector to plane containing both T_1 & T_2.
4) Obtain Eq. of plane (Normal + P_0).

$$1a)\ z - z_0 = f_y(x_0, y_0)(y - y_0)$$

$$1b)\ z - z_0 = f_x(x_0, y_0)(x - x_0)$$

$$2a)\ T_1 = \langle 0, 1, f_y \rangle$$

$$2b)\ T_2 = \langle 1, 0, f_x \rangle$$

$$N = T_1 \times T_2 = \begin{vmatrix} i & j & k \\ 0 & 1 & f_y \\ 1 & 0 & f_x \end{vmatrix} = \langle f_x, f_y, -1 \rangle$$

$$N \cdot (P - P_0) = f_x(x - x_0) + f_y(y - y_0) - 1(z - z_0) = 0$$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Can you approximate a function of two variables with a linear function?

Concept: linearization of a function at a point (x_0,y_0).

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

$$z = z_0 + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

$$z \approx f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

Problem: find tangent plane to given surface S at point P_0.
Solution: compute partial derivatives, evaluate them at P_0, plug in formula.

Q. What implicit assumption have we made above?
A. Function *f* has cont. 1^{st} partial derivatives.

Q. What happens if one or both partial derivatives are not cont.?

What happens if a function behaves badly?

Single variable function y = f(x):
Examples: section 2.8 p.159
A function *f(x)* is differentiable at *a* if *f'(a)* exists.
Theorem: If *f(x)* is differentiable at *a*, then it is cont. at a.
If *f(x)* is differentiable at *a,*

$$\Delta y = f(a+\Delta x) - f(a) = f'(a)\Delta x + \varepsilon \Delta x$$

$$where, \varepsilon \to 0 \quad as \quad \Delta x \to 0$$

$$dy = f'(x)dx$$

Reference: section 3.10, p.253, figure.5 & p.204
Concept: if function y = *f(x)* is differentiable, then let the differential be an independent variable *dx*.
Then d*y* represents the change in height of tangent line (change in linearization).
Δy (change in curve) might be hard to compute, *dy* is a good approx. close to a.

How is the differential defined for multivariate functions?

Two variable function z = f(x,y):

Define differentiability by analogy:

A function *f(x,y)* is differentiable at (*a,b)* if the partial derivatives exist near the point, and are cont. at (a,b).

If *f(x,y)* is differentiable at (*a,b),*

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

$$= f_x(a,b)\Delta x + \varepsilon_1 \Delta x + f_y(a,b)\Delta y + \varepsilon_2 \Delta y$$

$$where, \varepsilon_1, \varepsilon_2 \to 0 \quad as \quad (\Delta x, \Delta y) \to (0,0)$$

Concept: if function z = *f(x,y)* is differentiable, then let the differentials be independent variables *dx, dy*. The total differential dz is defined as:

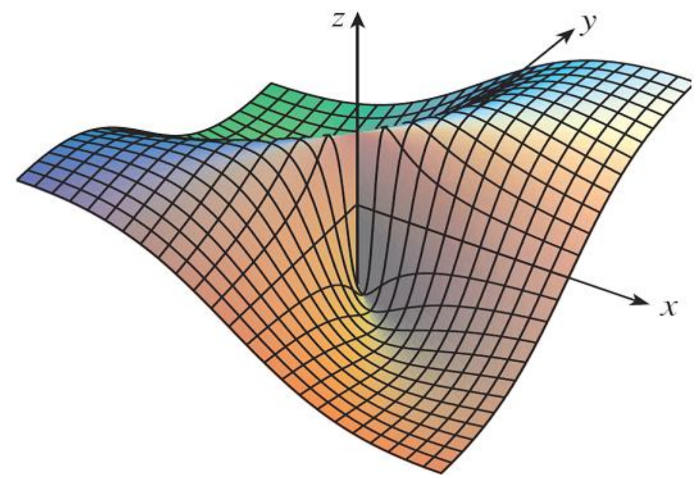
10

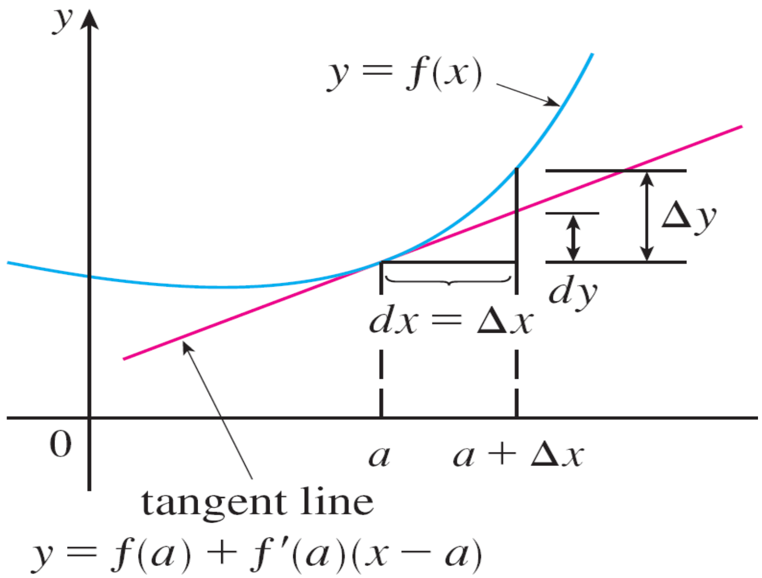
$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

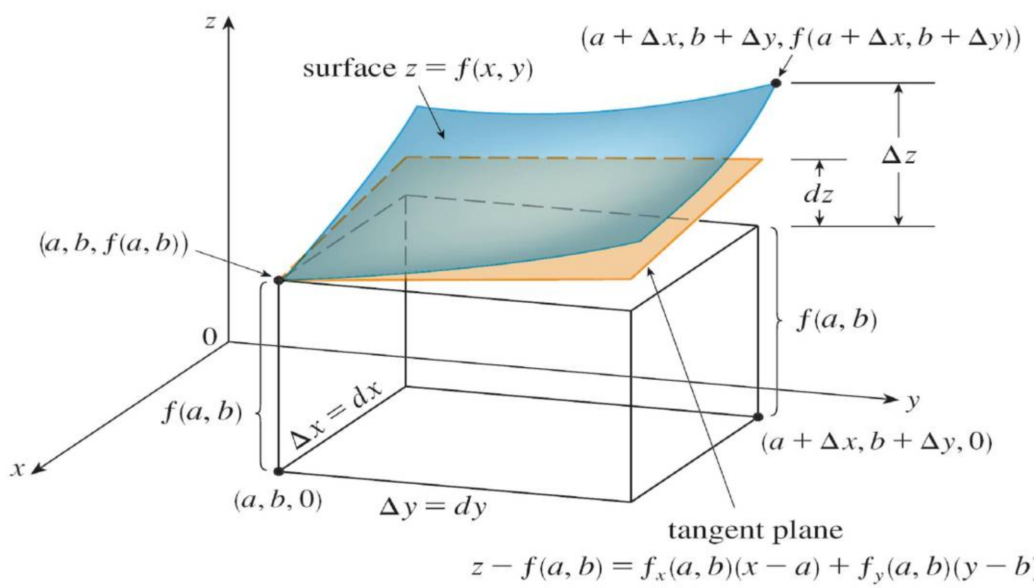
Then dz represents the change in height of tangent plane.

Δz (change in surface) might be hard to compute, *dz* is a good approx.

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$







Related Problems

Problem: show that a function z = *f(x,y)* is differentiable at a point P, and find its linearization there.
Solution: compute partial derivatives, if both cont. at point, *f(x,y)* is differentiable at P. To compute linearization, plug into formula.

Problem: given a function z = *f(x,y)* , find the differential dz.
Solution: compute partial derivatives and plug into formula 10.

Problem: given a function z = *f(x,y)*, if x changes from a-to-b, and y changes from c-to-d, compare the values of dz and Δz.
Solution:
a) Set x = a, y = b, dx = $\Delta x = b-a$, dy = $\Delta y = d-c$, and plug into formula for dz.
b) Compute $\Delta z = f(b,d) - f(a,c)$