Math S1201
Calculus 3
Chapters 13.2 – 13.4, 14.1

Summer 2015
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Outline

- CH 13.2 Derivatives of Vector Functions
  - Extension of definition
  - Tangent vector & 2^{nd} derivative
  - Differentiation rules
- CH 13.3 Arc Length & Curvature
  - Arc length of a curve in 2D (Review - Section 8.1, 10.2)
  - Arc length function, parametrization
  - Arc length for space curves
  - Integrals of vector functions
  - Curvature – general meaning
  - Curvature at any point of a function / space curve
  - Tangent, normal vectors
  - Derivation of formulae

Outline

- CH 13.4 Motion
  - Position, velocity, acceleration of a particle
  - Binormal vector, osculating circle
  - Components of acceleration
  - Derivation
- CH 14.1 Functions of Several Variables
  - Functions of n variables: definition
  - Visualizing functions – traces, level surfaces, colors

Guiding Eyes (13.2)

A. How is the derivative of a vector function defined?

B. Can all the differentiation rules be extended to vector functions?

What is the derivative of a vector function?

Recall definition of vector function in terms of components.
Consider the derivative with respect to the parameter (t):

$$\boldsymbol{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{r}'(t) = \lim_{h \to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}$$

Q. What is the geometric significance of this derivative?
A. r'(t) is the tangent vector to the curve (assuming it exists and is not zero)

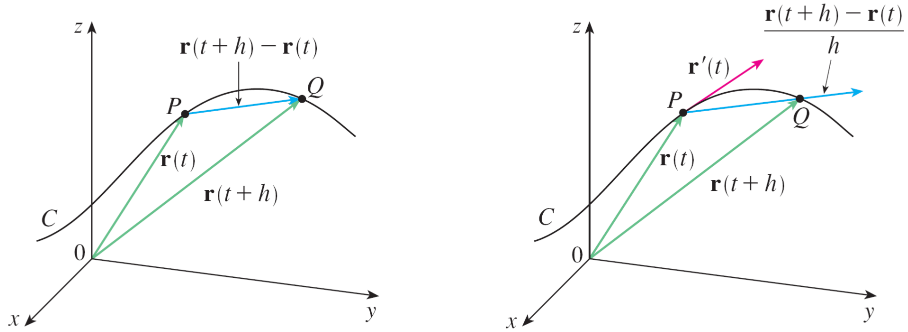
Q. Why can't the tangent vector be zero?
Q. What assumptions should we make about the component functions?

Concept: unit tangent vector

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$$

1

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$



2 Theorem If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

What is the derivative of a vector function?

Theorem: the derivative of a vector function is the defined as the derivative of each component separately.

$$\boldsymbol{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\boldsymbol{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Proof idea:

1) Use definition of LHS in terms of limits.
2) Re-write LHS in terms of components.
3) Apply limit to each component separately.

Note: result about limits of vector functions required.

$$\boldsymbol{r}'(t) = \lim_{h \to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}$$

$$= \lim_{h \to 0} \frac{\langle f_1(t+h), f_2(t+h), f_3(t+h) \rangle - \langle f_1(t), f_2(t), f_3(t) \rangle}{h}$$

$$= \left\langle \lim_{h \to 0} \frac{f_1(t+h) - f_1(t)}{h}, \ldots, \ldots \right\rangle = \langle f_1'(t), f_2'(t), f_3'(t) \rangle$$

Concept: the 2nd derivative is defined by analogy.

Can we extend differentiation rules to vector functions?

Yes. Proof approach:

1) Use definition to express LHS in terms of components.
2) Repeat with RHS.
3) Apply differentiation rules to components of LHS
4) Manipulate form to resemble RHS.

Related Problems

Problem: given a vector function, find the derivative and tangent vector at some point P. Find the tangent line at the same point.

Solution:

1) Compute derivative for each component separately.
2) Plug value of t corresponding to point P into the derivative (this gives the tangent vector at P).
3) The tangent line is parallel to the tangent vector at point P.

3 **Theorem** Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

Problem: prove that if every point on a (cont.) curve is equidistant from the origin, then the tangent vector at every point on the curve is orthogonal to the position vector.

Solution:

1) Assumption states that the length of the position vector is constant.
2) We want to show that position and tangent vectors are orthogonal.
3) Use definition of dot product and assumption.
4) Differentiate the dot product of the position vector with itself.
5) Use differentiation rule #4.

1) $\left|\boldsymbol{r}(t)\right| = c$

2) $\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = 0$

3) $\left|\boldsymbol{r}(t)\right| = \sqrt{\boldsymbol{r}(t) \cdot \boldsymbol{r}(t)} = c \Rightarrow \boldsymbol{r}(t) \cdot \boldsymbol{r}(t) = c^2$

4) $\dfrac{d}{dt}\left[\boldsymbol{r}(t) \cdot \boldsymbol{r}(t)\right] = \dfrac{d}{dt} c^2 = 0$

5) $\dfrac{d}{dt}\left[\boldsymbol{r}(t) \cdot \boldsymbol{r}(t)\right] = 2\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = 0$

Guiding Eyes (13.3)

A. How do you compute the length of a space curve?

B. What is the purpose of parametrizing a curve?

C. What is the curvature of a curve?

D. How do you derive the formulae for the curvature of a vector function?

Guiding Eyes (13.4)

A. How can you describe a particle's motion through space?

B. What is the interpretation of a particle's acceleration?

How do you compute the length of a curve?

Review (section 8.1): find length of curve defined by $y = f(x), a \le x \le b$
We assume that the curve is smooth (cont. derivatives) on the interval [a,b]

Q. How do you compute the circumference of a circle?
A. Limit of the circumference of inscribed polygons.

General approach:
1) For any curve, approximate curve with connected line segments.
2) Length of curve is approx. the sum of the segment lengths.
3) Take limit as # of segments goes to infinity.
Need to compute length of each line segment.

Q. Does the approach generalize to space curves?
A. Easier to answer if the curve is defined by parametric equations.

How do you compute the length of a curve?

General approach:

1) Length of line segment.
2) By MVT.
3) Substitute into length.
4) Take limit.

$$1)\ L_p = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$2)\ \exists c : f'(c) = \frac{\Delta y}{\Delta x} \Rightarrow \Delta y = f'(c)\Delta x$$

$$3)\ L_p = \sqrt{(\Delta x)^2 + (f'(c)\Delta x)^2}$$

$$= \sqrt{(\Delta x)^2 \left(1 + [f'(c)]^2\right)}$$

$$= \Delta x \sqrt{\left(1 + [f'(c)]^2\right)}$$

$$4)\ L = \lim_{\# \to \infty} L_p = \int_a^b \sqrt{\left(1 + [f'(x)]^2\right)} dx$$

Concept: arc length function denotes the distance along the curve from (a) to x ≤ b.

$$s(x) = \int_a^x \sqrt{\left(1 + [f'(t)]^2\right)} dt$$

How do you compute the length of a curve?

Review (section 10.2): find length of curve defined by parametric eqs.

$$x = f(t), y = g(t), a \le t \le b$$

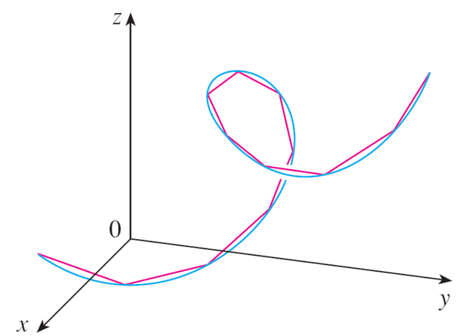
We assume that the curve is traversed once from left to right.

Q. What is the difference in the approach?

A. We express length in terms of Δt, hence we need to apply the MVT twice, to Δx and Δy for each line segment.

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

How do you compute the length of a space curve?



Generalize previous definition to 3D:

$$x = f(t), y = g(t), z = h(t), a \le t \le b$$

We assume curve is traversed once from left to right.

2

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$L = \int_a^b |\mathbf{r}'(t)|\, dt$$

Concept: the definite / indefinite integral of a cont. vector function is obtained by integrating each component.

What is the purpose of parametrizing a curve?

Problem: find the length of an arc given a vector function and two (start + end) points.

Solution:

1) Compute the derivative of the vector function.
2) Compute the length of the function from (1).
3) Integrate (2) with appropriate interval for the (t) parameter.

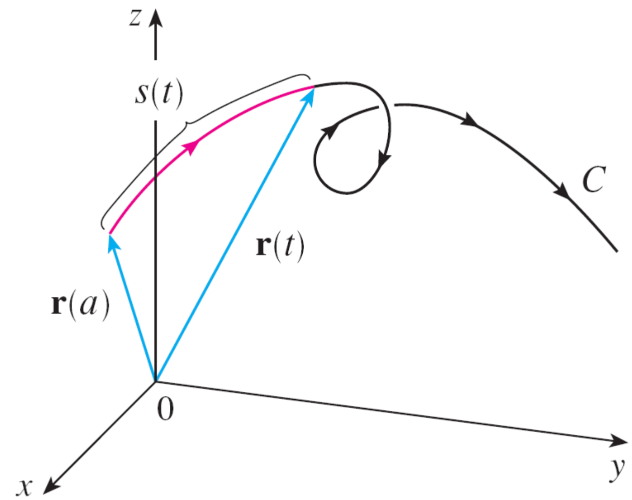
Concept: the parametrization of a space curve is some (different) representation of the same curve.

Concept: arc length parametrization for space curves.

$$s(x) = \int_a^x \sqrt{\left(1 + [f'(t)]^2\right)}dt \qquad \frac{ds}{dt} = |\mathbf{r}'(t)|$$

6

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$



6

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

What is the curvature of a curve?

Concept: a smooth curve has no sharp turns, corner or breaks.

Examples:

1) Straight line with slope zero: derivative is zero, curvature (κ) is zero.
2) Straight line with any slope: derivative is constant, still $\kappa = 0$.
3) Circle with radius r_1: curvature is constant $\kappa = C_1$.
4) Circle with radius $r_2 > r_1$: curvature is constant $\kappa = C_2 < C_1$.

Observe: curvature should be large if curve changes direction quickly (the larger the 2^{nd} derivative the larger the κ).

Q. What is the curvature at any point on a given function or space curve?

A. The function $\kappa(t)$ is a measure of how quickly the curve changes direction at point t.

Q. How can you compute the curvature at a point?

What is the curvature of a curve?

1) Assuming the curve is given by the vector function r(t).
2) The rate of change of the curve is given by the tangent vector r'(t).
3) If we only care about the direction of the tangent, consider the unit tangent vector T(t).
4) Curvature is defined as the magnitude of the rate of change of the unit tangent vector with respect to arc-length.
5) Quantity is easier to compute if expressed in terms of parameter (t).

$$1)\quad \boldsymbol{r}(t) \qquad 2)\quad \boldsymbol{r}'(t)$$

$$3)\quad \boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} \qquad 4)\quad \kappa(t) = \left|\frac{d\boldsymbol{T}}{ds}\right|$$

$$5)\quad \frac{d\boldsymbol{T}}{dt} = \frac{d\boldsymbol{T}}{ds}\frac{ds}{dt} \Rightarrow \frac{d\boldsymbol{T}}{ds} = \frac{d\boldsymbol{T}/dt}{ds/dt} = \frac{\boldsymbol{T}'(t)}{|\boldsymbol{r}'(t)|}$$

$$\Rightarrow \kappa(t) = \left|\frac{d\boldsymbol{T}}{ds}\right| = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|}$$

Particle's motion through space

Consider a particle moving through space.
1) Its position at time t is given by the vector function r(t).
2) We can approximate the direction of the particle at time t.
3) The limit of the approximation is the velocity vector v(t).
4) The magnitude of the velocity vector is the speed of the particle.
5) The acceleration is the rate of change of velocity, i.e. 2^{nd} derivative of r(t)

$$1) \quad \boldsymbol{r}(t) \quad 2) \quad \frac{\boldsymbol{r}(t+h)-\boldsymbol{r}(t)}{h} \quad 3) \quad \boldsymbol{v}(t)=\boldsymbol{r}'(t)$$

$$4) \quad v=|\boldsymbol{v}(t)|=|\boldsymbol{r}'(t)|=\frac{ds}{dt} \quad 5) \quad \boldsymbol{a}(t)=\boldsymbol{r}''(t)$$

Consider a person inside a car on a curving road.
The sharper the curve (curvature is large) and the higher the speed of the car, the faster the person is thrown against the door.
In other words, acceleration occurs in a particular direction.

Motion through space: acceleration

1) The unit normal vector **N** indicates the direction in which the curve is turning at each point.
2) Acceleration can be resolved into 2 components (tangential and normal).

$$\boldsymbol{T}(t)=\frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}=\frac{\boldsymbol{v}}{v} \Rightarrow \boldsymbol{v}=v\boldsymbol{T}$$

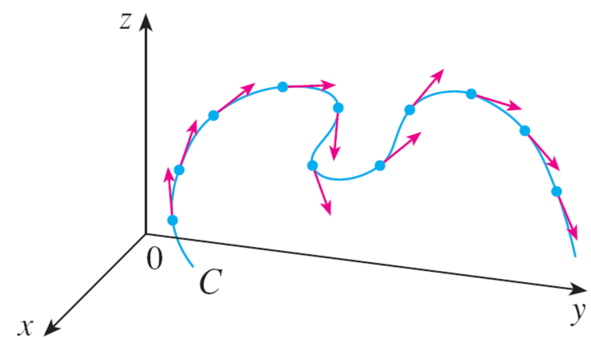
$$\boldsymbol{a}(t)=\boldsymbol{r}''(t)=\boldsymbol{v}'=v'\boldsymbol{T}+v\boldsymbol{T}'$$

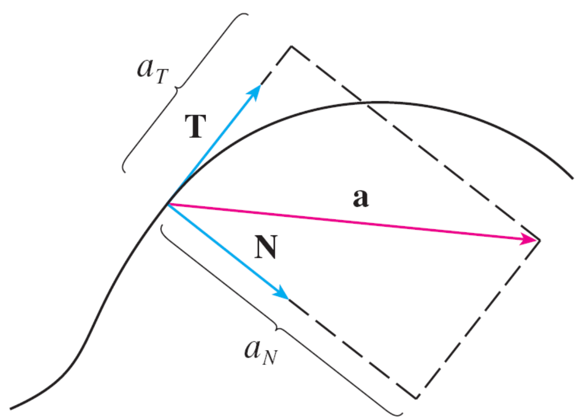
$$\forall t, |\boldsymbol{T}(t)|=1 \Rightarrow \boldsymbol{T}(t)\cdot\boldsymbol{T}'(t)=0$$

$$\boldsymbol{N}(t)=\frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|} \Rightarrow \boldsymbol{T}'=\boldsymbol{N}|\boldsymbol{T}'|$$

$$\kappa(t)=\frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|}=\frac{|\boldsymbol{T}'|}{v} \Rightarrow |\boldsymbol{T}'|=\kappa v$$

$$\boldsymbol{a}=v'\boldsymbol{T}+v|\boldsymbol{T}'|\boldsymbol{N}=v'\boldsymbol{T}+\kappa v^2\boldsymbol{N}$$





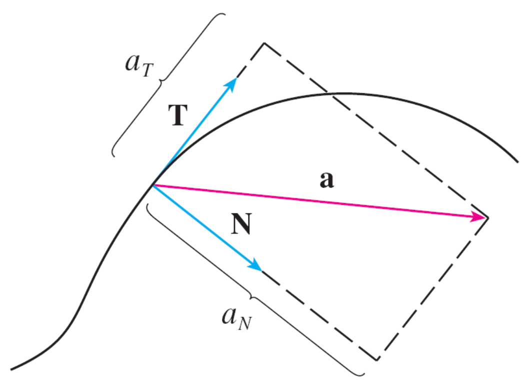
Motion through space: binormal

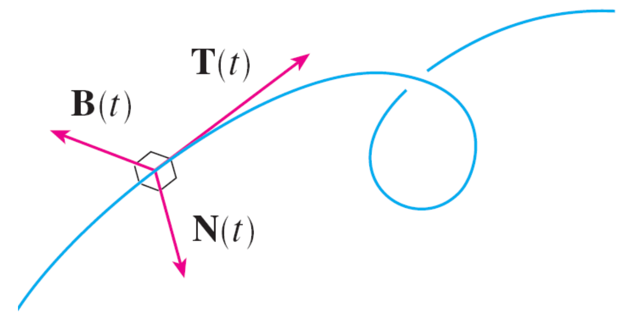
Equation for acceleration confirms intuition: if we increase curvature/speed, normal component of acceleration increases.
Observe: no matter how object moves through space, its acceleration vector is always in the osculating (kissing) plane containing **T** and **N**.

Concept: the osculating circle in the direction of **N** with radius 1/κ
Concept: the binormal vector defines a “B-N-T” frame.
Concept: the normal plane is defined by the **N, B** vectors and contains vectors orthogonal to the tangent.

$$\boldsymbol{a} = v'\boldsymbol{T} + \kappa v^2 \boldsymbol{N}$$

$$\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$$





Formulae for curvature

Q. Can we express κ(t) **in terms of r'(t), r''(t) vectors only?**
A. To accomplish this, we need to express **T'(t)**.

Q. Why do we need this new formula?
A. Easier to compute curvature in practice.

Q. What about the special case where y = f(x)?
A. The formula we derive can be simplified.

Q. Can we express components of acceleration in terms of r(t), r'(t), r''(t)?
A. To accomplish this, need to express v' (derivative of speed function).

$$\kappa(t) = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{|\boldsymbol{T}'|}{v} \qquad \boldsymbol{a} = v'\boldsymbol{T} + \kappa v^2 \boldsymbol{N}$$

$$\boldsymbol{T} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|} = \frac{\boldsymbol{r}'}{v} \quad \kappa(t) = \left|\frac{d\boldsymbol{T}}{ds}\right| = \frac{|\boldsymbol{T}'|}{|\boldsymbol{r}'|} = \frac{|\boldsymbol{T}'|}{v} \quad |\boldsymbol{T}| = 1 \;\Rightarrow\; \boldsymbol{T}\cdot\boldsymbol{T}' = 0$$

$$\boldsymbol{T}' = \frac{\boldsymbol{r}''}{|\boldsymbol{r}'|} - \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|^2}\frac{d}{dt}|\boldsymbol{r}'| = \frac{\boldsymbol{r}''}{v} - \frac{\boldsymbol{r}'}{v^2}v'$$

$$|\boldsymbol{T}\times\boldsymbol{T}'| = |\boldsymbol{T}||\boldsymbol{T}'|\sin\theta_{T,T'} = |\boldsymbol{T}||\boldsymbol{T}'|\sin\frac{\pi}{2} = |\boldsymbol{T}||\boldsymbol{T}'| = |\boldsymbol{T}'|$$

$$\boldsymbol{T}\times\boldsymbol{T}' = \boldsymbol{T}\times\left(\frac{\boldsymbol{r}''}{v} - \frac{\boldsymbol{r}'}{v^2}v'\right) = \left(\boldsymbol{T}\times\frac{\boldsymbol{r}''}{v}\right) - \left(\boldsymbol{T}\times\frac{\boldsymbol{r}'}{v^2}v'\right)$$

$$= \left(\frac{\boldsymbol{r}'}{v}\times\frac{\boldsymbol{r}''}{v}\right) - \left(\frac{\boldsymbol{r}'}{v}\times\frac{\boldsymbol{r}'}{v^2}v'\right) = \frac{\boldsymbol{r}'\times\boldsymbol{r}''}{v^2}$$

$$|\boldsymbol{T}\times\boldsymbol{T}'| = |\boldsymbol{T}'| = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{v^2} = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^2}$$

$$\kappa(t) = \frac{|\boldsymbol{T}'|}{|\boldsymbol{r}'|} = \frac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}$$

$$y = f(x) \Rightarrow \boldsymbol{r}(x) = \left\langle x, f(x) \right\rangle$$

$$\boldsymbol{r}'(x) = \left\langle 1, f'(x) \right\rangle \quad \boldsymbol{r}''(x) = \left\langle 0, f''(x) \right\rangle \quad \left|\boldsymbol{r}'(x)\right| = \sqrt{1 + \left[f'(x)\right]^2}$$

$$\boldsymbol{r}'(x) \times \boldsymbol{r}''(x) = \begin{vmatrix} i & j & k \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x)\boldsymbol{k}$$

$$\left|\boldsymbol{r}'(x) \times \boldsymbol{r}''(x)\right| = \left|f''(x)\boldsymbol{k}\right| = \left|f''(x)\right|$$

$$\kappa(x) = \frac{\left|\boldsymbol{r}'(x) \times \boldsymbol{r}''(x)\right|}{\left|\boldsymbol{r}'(x)\right|^3} = \frac{\left|f''(x)\right|}{\left[1 + \left(f'(x)\right)^2\right]^{3/2}}$$

$$\boldsymbol{a} = v'\boldsymbol{T} + \kappa v^2 \boldsymbol{N} \qquad \boldsymbol{v} = v\boldsymbol{T}$$

$$\kappa = \frac{|\boldsymbol{T}'|}{|\boldsymbol{r}'|} = \frac{|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}$$

$$\boldsymbol{v} \cdot \boldsymbol{a} = v\boldsymbol{T} \cdot \left(v'\boldsymbol{T} + \kappa v^2 \boldsymbol{N} \right) = vv'\boldsymbol{T} \cdot \boldsymbol{T} + \kappa v^3 \boldsymbol{T} \cdot \boldsymbol{N} = vv'$$

$$v' = \frac{\boldsymbol{v} \cdot \boldsymbol{a}}{v} = \frac{\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t)}{|\boldsymbol{r}'(t)|} \qquad \kappa v^2 = \frac{|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Related Problems

Problem: given a circle with radius **a**, show that κ = 1/a.

Solution:

1) Choose origin as center of circle.
2) Parametrize eq. (t for angle).
3) Differentiate, compute magnitude.
4) Plug into formula.

$$\boldsymbol{r}(t) = \langle a\cos t, a\sin t \rangle$$

$$\boldsymbol{r}'(t) = \langle -a\sin t, a\cos t \rangle$$

$$|\boldsymbol{r}'(t)| = a$$

Problem: given the position vector of a particle. Find its velocity, speed, and acceleration.

Solution: differentiate vector function (twice), and compute magnitude.

Problem: given acceleration vector and initial conditions (velocity and position at some time t). Find the velocity and position vectors.

Solution:

1) Integrate acceleration, determine constant vector from initial condition.
2) Integrate velocity, determine constant vector from initial condition.

In general:

$$\boldsymbol{v}(t) = \boldsymbol{v}(t_0) + \int_{t_0}^{t} \boldsymbol{a}(u)\,du$$

Guiding Eyes (14.1)

A. Functions of several variables – definition

B. How do you visualize a function of two (3) variables?

Functions of several variables

Concept: a function *f* of two variables is a rule that assigns each ordered pair of real numbers (x,y) in the domain of *f*, a unique real number denoted by *f*(x,y). The set of all values *f* can take is the range of *f*.

Q. How is a function of n variables defined?

Problem: find the domain and range of a function *f*(x,y).
Solution: domain includes all values where function is defined. Range includes all valid values for z = *f*(x,y).

Wind speed (km/h); rows: Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

How do you visualize a function of 2 variables?

Concept: the graph of a function consists of all triples (x,y,z) such that (x,y) are in the domain of *f* and z = *f*(x,y).

Examples:

1) Linear function (plane).
2) Half sphere.
3) Elliptic paraboloid.

$$z = ax + by + c$$

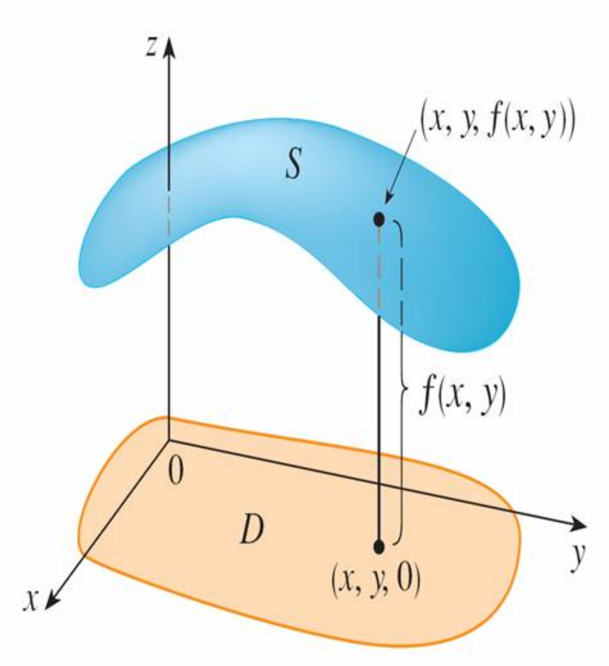
$$z = f(x,y) = \sqrt{9 - x^2 - y^2}$$

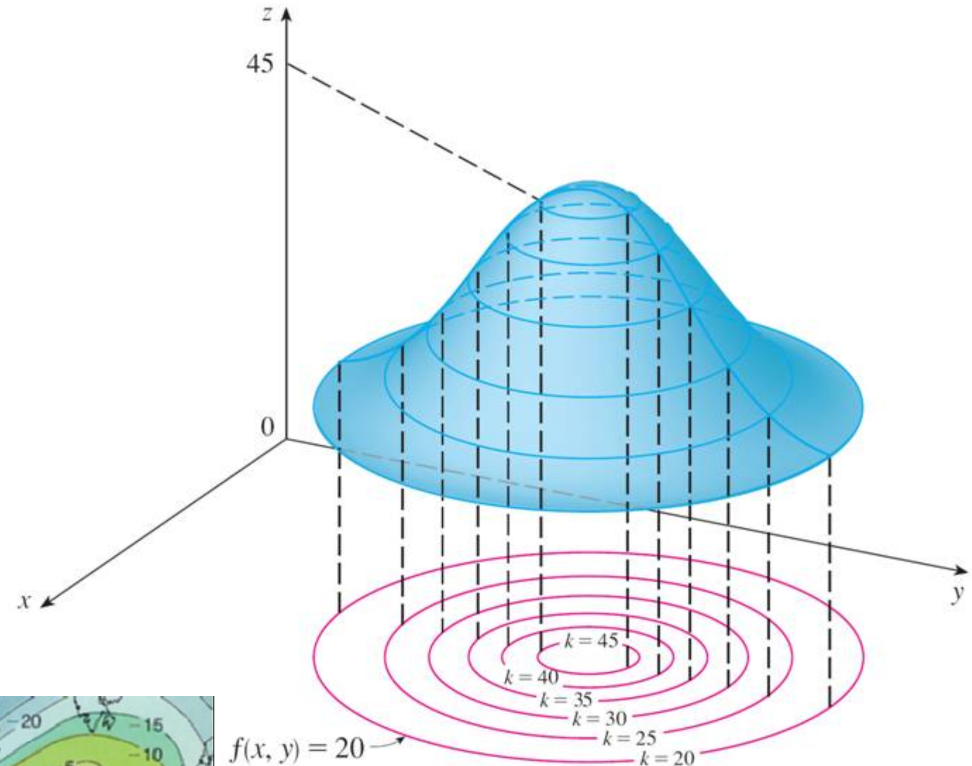
$$z = f(x,y) = 4x^2 + y^2$$

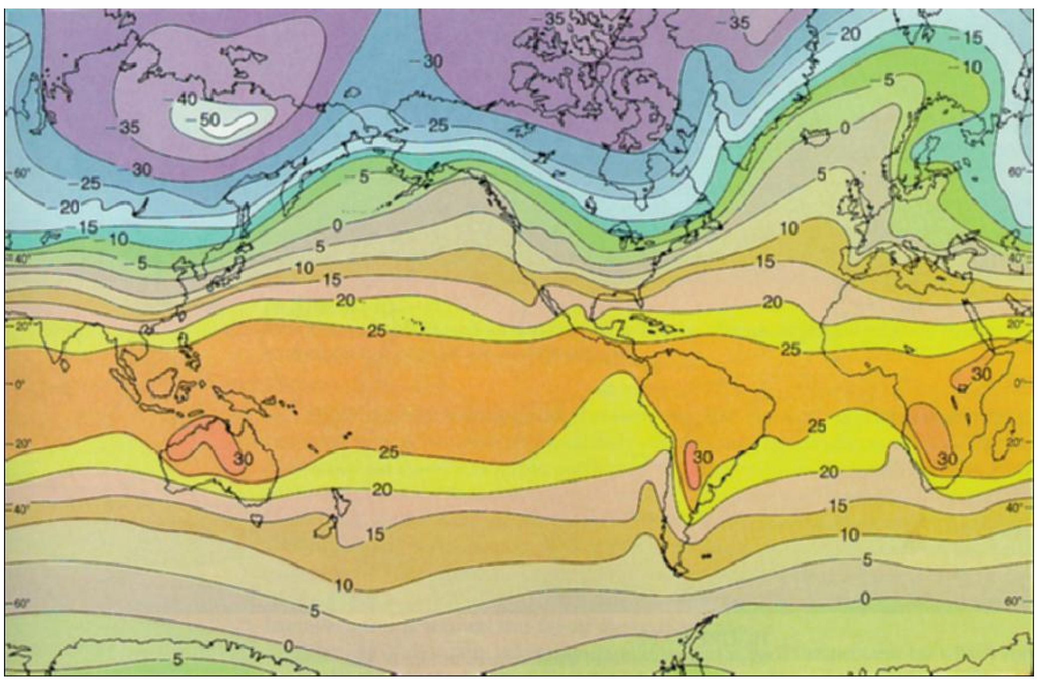
Concept: the level curves (contour lines) of a function are defined by fixing a constant: *f*(x,y) = k. For a function of 3 variables, we have level surfaces.

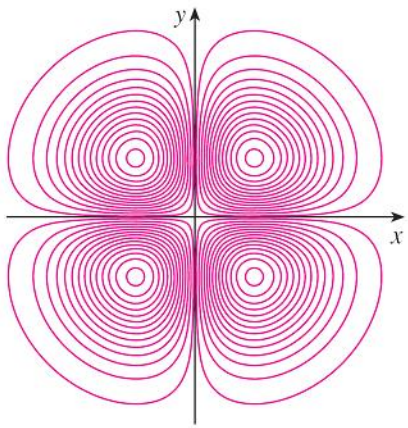
Q. How do you sketch a function of two variables?
A. Draw level curves on plane, lift up to indicated height. Colors can help (red = hottest = highest values; blue = coldest = smallest).

Problem: sketch the level curves of a function *f*(x,y).
Solution: draw level curves for a chosen set of values at an appropriate interval. Altogether, the curves create a contour map.

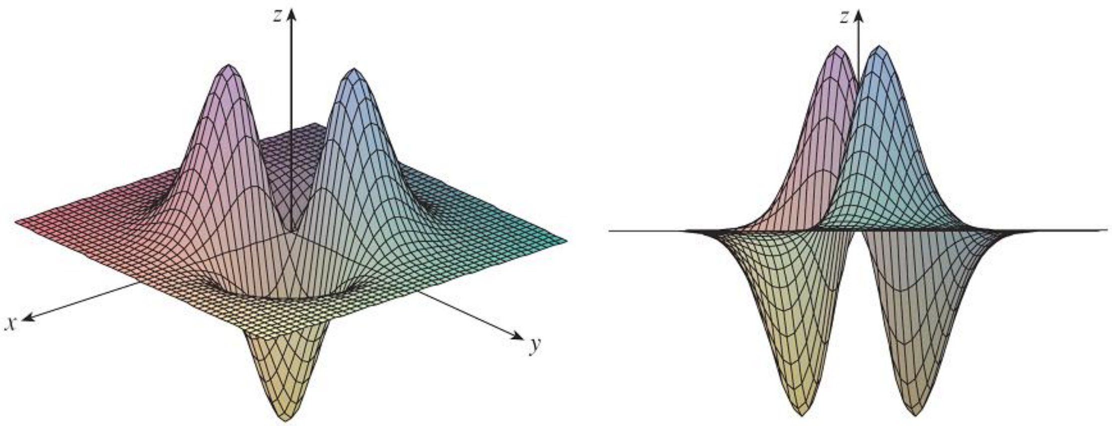

z
S
(x, y, f(x, y))
f(x, y)
0
D
(x, y, 0)
x
y


z
45
0
x
y
k = 45
k = 40
k = 35
k = 30
k = 25
k = 20
f(x, y) = 20

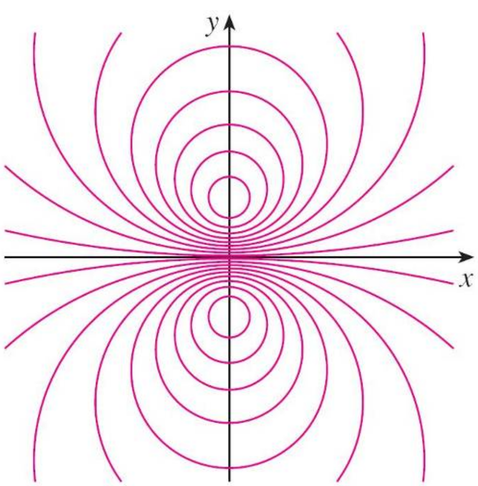




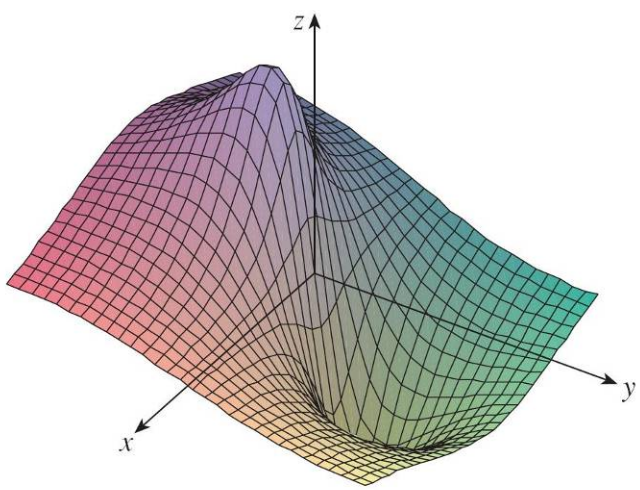
(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$



(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$



Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$



$f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$