Math S1201
Calculus 3
Chapters 12.5 – 12.6, 13.1

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Outline

- CH 12.5 Lines and Planes
  - Equation of line in 3D
  - Parametric & symmetric equations
  - Line segments and skew lines
  - Equation of plane
  - Normal vector
  - Intersection of planes
  - Point-2-plane distance
  - Line-2-line distance

Outline

- CH 12.6 Surfaces
  - Conic sections (review)
  - Quadratic surfaces
  - Traces
  - Sketching & identifying surfaces
- CH 13.1 Vector Functions
  - Curves
  - Vector functions
  - Limits & continuity of vector functions

<u>Guiding Eyes (12.5)</u>

A. What is the (general) equation of a line in 3D?

B. What is the (general) equation of a plane in 3D?

C. What can we tell about the intersection of two planes (line and plane)?

D. What is the distance between a point and a plane (two planes, two skew lines)?

What is the equation of a line in 3D?

Q. What is the equation of a line in 2D?
A. two familiar forms: slop-intercept & point-slope

$$(1)\ y = mx + b \qquad (2)\ y - y_0 = m(x - x_0)$$

Q. Does the equation generalize to 3D?
A. No! Equation of plane. What we really want is:

$$y - y_0 = m(x - x_0) \quad and \quad z = z_0$$

Different approach: define a line as an arbitrary point **P(x_0,y_0)** and a vector **v = < a, b >** representing the direction. Introduce a parameter (**t**) and obtain a vector function:

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v} \qquad \langle x, y \rangle = \langle x_0 + ta, y_0 + tb \rangle$$

Concept: parametric equations $\quad x = x_0 + ta \quad y = y_0 + tb$

Easy to generalize to 3D!

What is the equation of a line in 3D?

Problem: find the parametric equation for a line passing through point **P**, where the line is parallel to some vector **v**.
Solution: **P** defines the position vector $\mathbf{r}_0$, **v** defines direction. Plug in.

Problem: find point(s) on a line given by parametric equations.
Solution: set the parameter (t) to some value(s)

Concept: symmetric equations

$$x = x_0 + ta \quad y = y_0 + tb \quad z = z_0 + tc$$

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Problem: find the equation of a line that passes through two points **P, Q**.
Solution: compute direction vector **v = < a, b, c >** , use one point as the position vector $\mathbf{r}_0$. Plug in.

What is the equation of a line in 3D?

Concept: if we restrict the range of the parameter (t in [0,1]), we obtain a line segment.

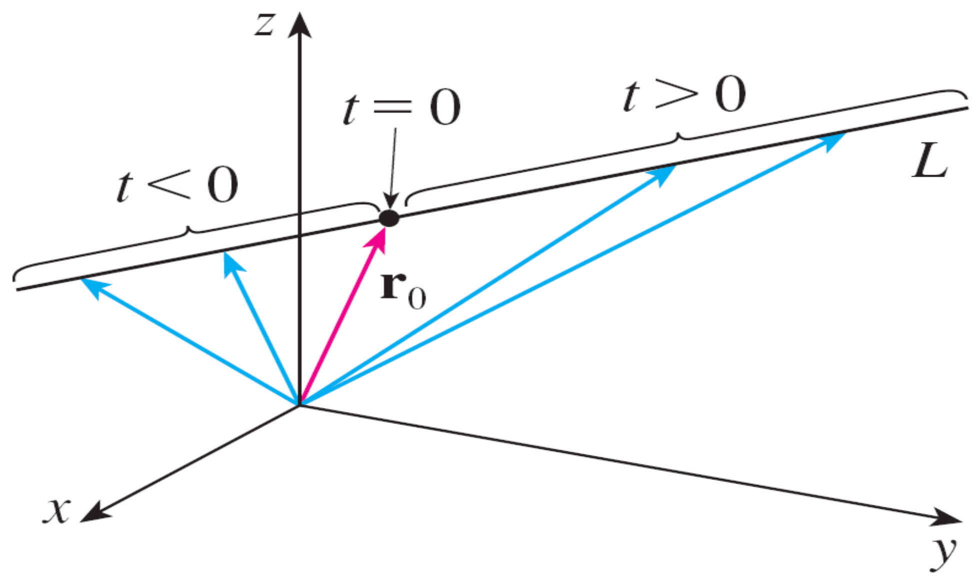
$$\boldsymbol{v} = \boldsymbol{r}_1 - \boldsymbol{r}_0 \Rightarrow \boldsymbol{r}(t) = \boldsymbol{r}_0 + t\left(\boldsymbol{r}_1 - \boldsymbol{r}_0\right) = (1-t)\boldsymbol{r}_0 + t\boldsymbol{r}_1$$

Q. Lines in 2D either intersect or they are parallel. What about 3D?
A. We have an extra "degree of freedom", hence a 3rd possibility, skew lines.

Problem: given the equations of two lines, show that these are skew lines.
Solution: 1) If the direction vectors for the lines are not parallel, lines are not parallel (note: using cross product might be overkill).
2) For the lines to intersect, must find two values for the unknowns (t_1,t_2) to satisfy all 3 equations.

$$L1 : \langle x, y, z \rangle = \langle x_1, y_1, z_1 \rangle + t_1 \langle a_1, b_1, c_1 \rangle$$

$$L2 : \langle x, y, z \rangle = \langle x_2, y_2, z_2 \rangle + t_2 \langle a_2, b_2, c_2 \rangle$$



4 The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

What is the general equation of a plane?

General eq. of line in 2D: $y = mx + b \Rightarrow ax + by + c = 0$

Observe that in 3D, y = C, z = C, y = x are all equations of planes.

We know that a plane is linear.

"Guess" equation of plane to be: $ax + by + cz + d = 0$

This is the linear equation of a plane.

1) Plug in an arbitrary point $P(x_0,y_0,z_0)$ on that plane into equation.
2) Subtract from linear equation of plane.
3) Recognize the new equation as a dot product of two vectors.

4,5) This is the vector equation of a plane.

5) The first vector is called the normal vector to the plane.

$$ax_0 + by_0 + cz_0 + d = 0$$

$$ax + by + cz + d - (ax_0 + by_0 + cz_0 + d) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\langle a,b,c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

$$\mathbf{n} = \langle a,b,c\rangle \quad \mathbf{r} - \mathbf{r}_0 = \langle x - x_0, y - y_0, z - z_0\rangle \quad \mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0$$

What is the general equation of a plane?

Problem: find the equation for a plane which contains point **P**, and has normal vector **n**.
Solution: **P** defines vector $\mathbf{r}_0$. Plug into vector equation.

Problem: find the equation for a plane which contains three points **P,Q ,R**.
Solution: 1) Compute two vectors: PQ & PR.
2) Compute the normal vector to the plane (cross product of PQ, PR).
3) Choose any point, and together with normal, plug into vector equation.

Problem: find the point of intersection between a line and a plane.
Solution: 1) Assuming line is given by parametric eqs, plug these eqs into the linear eq of a plane and solve for the parameter (t).
2) Substitute the value of t into the line eq to obtain the point.

Concept: parallel planes have parallel normal vectors.

Intersection of two planes

Concept: the angle between two (non-parallel) planes is the acute angle between their normal vectors.

Problem: given the equations for two planes find the angle between them.
Solution: 1) Deduce the normal vectors of the planes from the eqs.
2) Compute cosine of angle between normals using dot product formula.

Problem: given the equations for two planes determine the equation of the line specified by their intersection.
Solution A: 1) Since the line of intersection lies in both planes, it is orthogonal to both normal vectors. Compute cross product of normal vectors. That is the direction vector of the line.
2) Find a point on the line of intersection (a point on both planes).

What is the drawback of this solution / approach?

Intersection of two planes

Problem: given the equations for two planes determine the equation of the line specified by their intersection.

Solution B:

1) Use plane eqs to express two variables (x,y) in terms of the third (z).

2) Choose parameter (t): z = t. We now have the paramertric eqs for the line (where we have assumed that the point on the line has $z_0 = 0$).

$$a_1 x + b_1 y + c_1 z + d_1 = 0 \rightarrow a_1 a_2 x + a_2 b_1 y + a_2 c_1 z + a_2 d_1 = 0$$

$$a_2 x + b_2 y + c_2 z + d_2 = 0 \rightarrow a_1 a_2 x + a_1 b_2 y + a_1 c_2 z + a_1 d_2 = 0$$

$$(a_2 b_1 - a_1 b_2) y + (a_2 c_1 - a_1 c_2) z + (a_2 d_1 - a_1 d) = 0$$

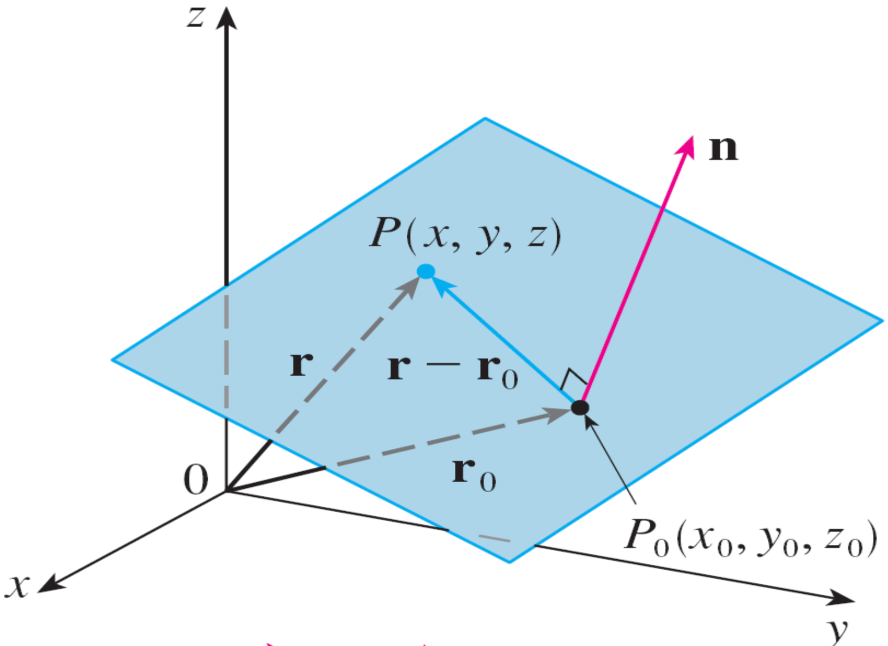
$$y = \frac{E_2 z + E_3}{-E_1} \Rightarrow a_1 x + b_1 \frac{E_2 z + E_3}{-E_1} + c_1 z + d_1 = 0$$

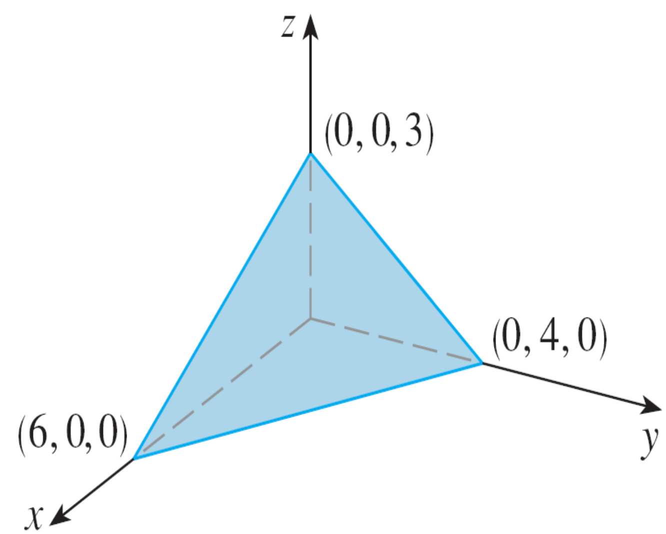
$$a_1 x = \frac{b_1 E_2 z + b_1 E_3 - c_1 E_1 z - d_1 E_1}{E_1}$$

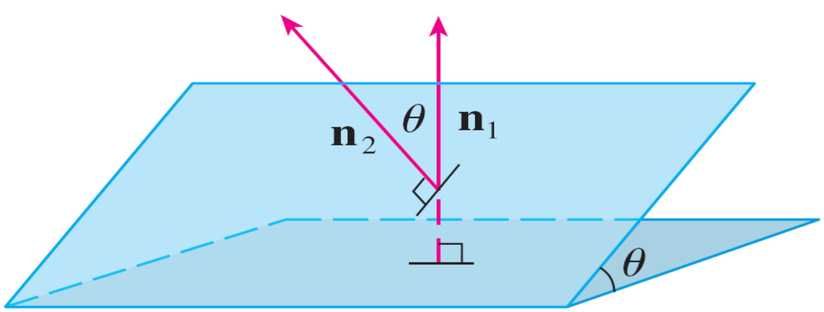
$$x = \frac{(b_1 E_2 - c_1 E_1) z + (b_1 E_3 - d_1 E_1)}{a_1 E_1}$$

$$ax + by + cz + d = 0$$

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$







What is the distance between a point and a plane?

Start with 2D, consider the distance from an any point P to any line.
The distance can be described by a vector (v) from P to a point Q(x,y) which satisfies the equation of the line.
Only we want the "shortest" distance, hence we compute the scalar projection of **v** onto the normal vector **n**.
Generalize idea to 3D: replace line with plane and add component to vectors.

$$\mathbf{n} = \langle a, b, c \rangle \quad \mathbf{r} - \mathbf{r}_0 = \langle x - x_0, y - y_0, z - z_0 \rangle \quad \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

$$\boldsymbol{b}_{sp} = |\boldsymbol{b}| \cos\theta = \frac{\mathbf{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|} = \frac{|\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0)|}{|\mathbf{n}|} = \frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Q. Why do we need absolute value?
A. Must take sign of normal vector into account!

Related Problems

Problem: given the eqs of two parallel planes, find the distance between them.

Solution:

1) Choose one point (P) on one (any) of the planes.
2) Compute distance to other plane using formula.

Problem: given the eqs of two skew lines, find the distance between them.

Solution: reduce the problem to a problem we have solved already

1) Since the lines are skew, they can be embedded into two parallel planes.
2) Parallel planes have a common normal vector which must be orthogonal to both lines.
3) The normal is the cross-product of the direction vectors of the lines
4) Find a point on one (any) line.
5) Obtain equation of plane (you have point and normal).
6) Find a point on the second line.
7) Now we can solve the previous problem with the plane eq (5) & point (6)

Guiding Eyes (12.6)

A. What are the simplest non-linear (quadratic) surfaces?

B. How do you sketch and identify a surface?

What are the simplest non-linear surfaces?

Q. What are the simplest non-linear curves in 2D?

A. Conic sections (review)

1) Parabola
2) Ellipse
3) Hyperbola
4) Shifted (translation)

$$(1)\ y = Cx^2 \qquad x = Cy^2$$

$$(2)\ \frac{x^2}{C_1} + \frac{y^2}{C_2} = 1$$

$$(3)\ \frac{x^2}{C_1} - \frac{y^2}{C_2} = 1 \qquad \frac{y^2}{C_1} - \frac{x^2}{C_2} = 1$$

$$(4)\ x \rightarrow x - h \qquad y \rightarrow y - k$$

By analogy, in 3D,
the simplest surfaces are quadratic.

Concept: a quadratic surface is the graph of a 2nd degree eq in 3 variables.

Given a quadratic eq, we can use rotation & translation to modify the eq to a well known, standard form.

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad Ax^2 + By^2 + Iz = 0$$

How do you sketch and identify a surface?

Q. How do you sketch a surface?
A. Consider traces (cross sections) of the surface.

Concept: traces (cross sections) are the curves of intersection of the surface with planes which are parallel to the coordinate planes.

Examples:

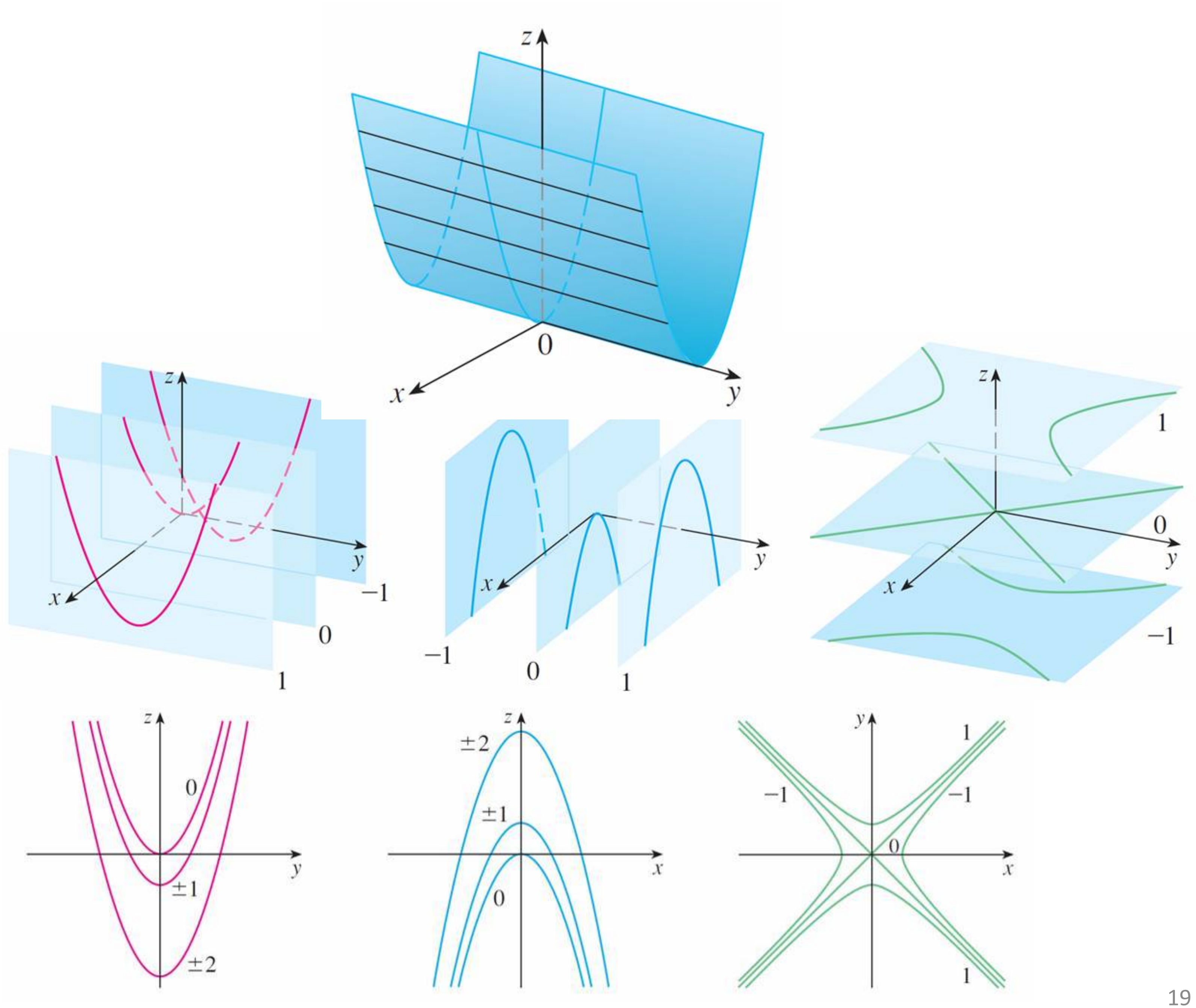
1) traces: parabolas, parallel to y-axis
2) traces: circles, parallel to z-axis
3) traces: circles, parallel to x-axis
4) traces in horizontal plane z = k: ellipses
   traces in xz, yz planes (y=0, x=0 are hyperbolas)

$$(1)\ x^2 - z = 0$$

$$(2)\ x^2 + y^2 = 1$$

$$(3)\ y^2 + z^2 = 1$$

$$(4)\ \frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$$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How do you sketch and identify a surface?

Q. How do you identify a surface?

Common quadratic surfaces:

1) Cylinder: surface consisting of all lines that are parallel to a given line and pass through a given plane curve.

2) Ellipsoid: traces (x,y,z=k) are ellipses. Symmetric with respect to coordinate planes (even powers only). Resembles a sphere.

3) Elliptic paraboloid: traces (x,y=k) are parabolas, and (z=k) ellipses.

4) Hyperbolic paraboloid: traces (x,y=k) are parabolas, & (z=k) hyperbolas.

Problem: given a quadratic eq, identify the surface.

Solution:

1) Convert to standard form: / or * by const
   complete squares, identify shifts & rotations.

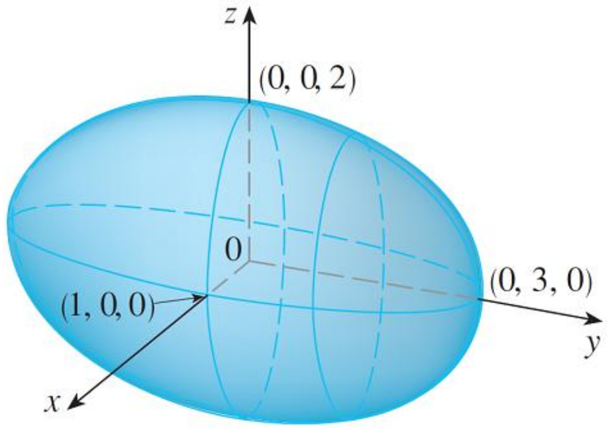
2) Use traces to prepare the sketch of the surface.

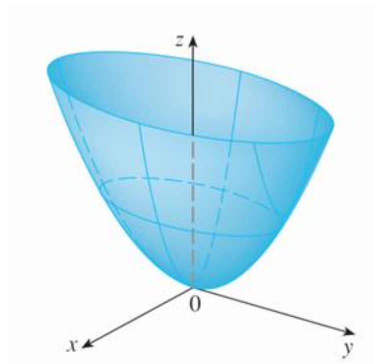
3) Recognize a familiar surface type.

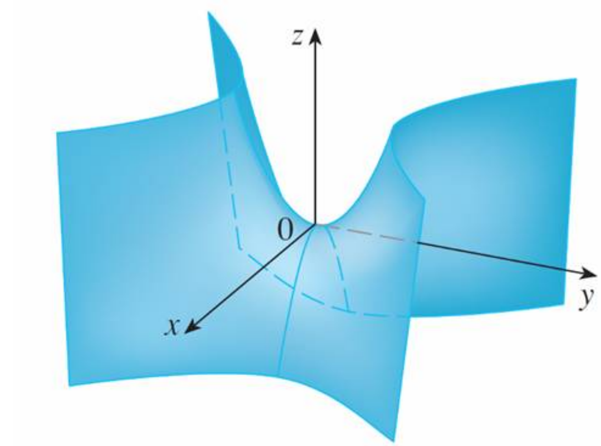
$$(2)\ \frac{x^2}{c_1}+\frac{y^2}{c_2}+\frac{z^2}{c_3}=1$$

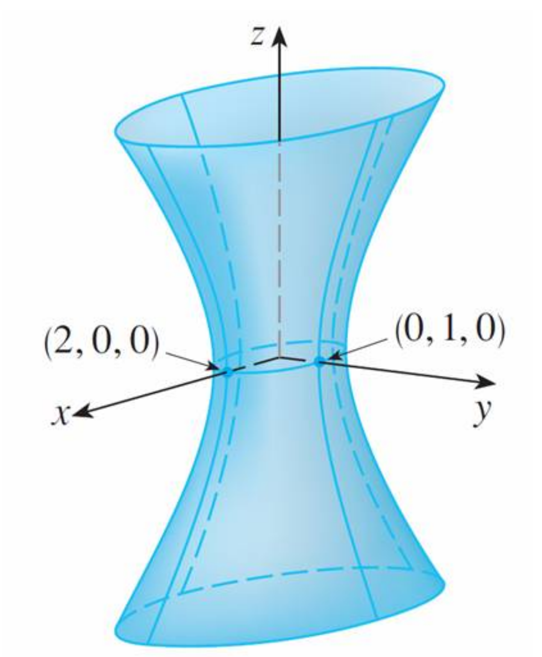
$$(3)\ \frac{x^2}{c_1}+\frac{y^2}{c_2}=\frac{z}{c_3}$$

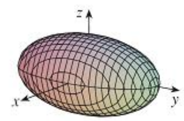
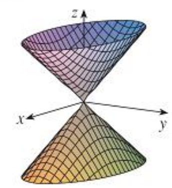
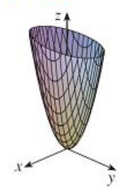
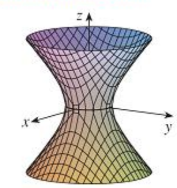
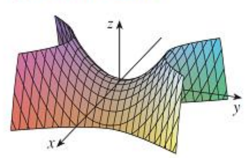
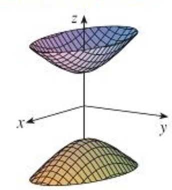
$$(4)\ \frac{x^2}{c_1}-\frac{y^2}{c_2}=\frac{z}{c_3}$$









Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Guiding Eyes (13.1)

A. **Can you generalize the equations of a line in space to describe any curve?**

B. **How are the concepts of limit and continuity defined for vector functions (space curves)?**

C. **How can you visualize space curves?**

What is the equation of a space curve?

Recall: we have parametric equations and vector eq for a line in space.
We can replace a linear function of **t** with any function.
Generalization to describe any space curve.

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + t\boldsymbol{v} \quad \longrightarrow \quad \boldsymbol{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Q. What implicit assumptions are we making?
A. We assume that {f,g,h} are continuous functions defined on an interval.

Q. What is the range of a vector function r(t)**?**
A. A set of vectors

Q. Can we introduce a physical analogy to a vector function?
A. The curve is traced out by a moving particle whose position at time **t** is **r**(t) = < f(t), g(t), h(t) >

How are limit and continuity defined for vector functions?

Idea: by analogy to vectors, consider each component separately.
Note: this approach still needs to be justified!
Concept: limit of a vector function is defined as the vector of limits of each component.

$$\boldsymbol{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\lim_{t \to a} \boldsymbol{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

Concept: a vector function r(t) is continuous (cont.) at a point (t = a) if the limit exists at (t=a) and is equal to r(a).
Note: it is cont. (at t=a) if all of its component functions are cont. (at t=a).

Problem: find the limit of a given vector function as t --> a.
Solution: find the limit as t --> a of each component separately.

Q. How do we prove that the limit of a vector function is correct?

How to prove that limit of vector function is correct?

Recall epsilon – delta definition of limit for function of a single variable.

$$\lim_{x\to a} f(x) = L \quad IF$$

$$\forall \varepsilon > 0, \exists \delta > 0 \ : \ 0 < |x-a| < \delta \to |f(x) - L| < \varepsilon$$

$$\lim_{t\to a} \boldsymbol{r}(t) = \mathbf{b} \quad IF$$

$$\forall \varepsilon > 0, \exists \delta > 0 \ : \ 0 < |t-a| < \delta \to |\boldsymbol{r}(t) - \mathbf{b}| < \varepsilon$$

Q. Can we justify the definition of vector function limit?

Proof: 1) Rewrite expression by definition of vector length.

2) Simplify and conclude inequality holds for each component (term).

3) By definition, limit for each component holds.

$$\lim_{t\to a} \boldsymbol{r}(t) = \left\langle \lim_{t\to a} f_1(t), \lim_{t\to a} f_2(t), \lim_{t\to a} f_3(t) \right\rangle$$

$$|\boldsymbol{r}(t) - \mathbf{b}| < \varepsilon \Rightarrow \sqrt{\sum_{k=1}^{n} \left(f_k(t) - b_k\right)^2} < \varepsilon$$

$$\sum_{k=1}^{n} \left(f_k(t) - b_k\right)^2 < \varepsilon^2 \Rightarrow \forall k, \left(f_k(t) - b_k\right)^2 < \varepsilon^2$$

$$\forall k, \sqrt{\left(f_k(t) - b_k\right)^2} < \varepsilon \Rightarrow \forall k, |f_k(t) - b_k| < \varepsilon$$

How can you visualize space curves?

Problem: given two points (P,Q), find the vector eq of the line segment joining the points.
Solution: use definition of line segment, where

$$\boldsymbol{r}(t) = (1-t)\boldsymbol{r}_0 + t\boldsymbol{r}_1 = (1-t)P + tQ,\ \ 0 \le t \le 1$$

Problem: find the vector function that represents the curve of intersection between two surfaces.
Sol: choose proper parametrization.

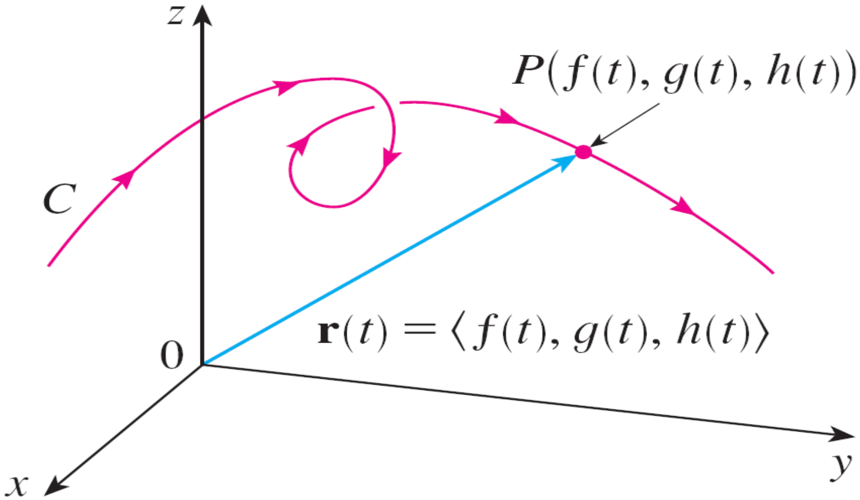
$$\text{Intersection: } x^2 + y^2 = 1,\ \ y + z = 2$$

$$y = \sin(t),\ x = \cos(t),\ 0 \le t \le 2\pi$$

$$z = 2 - y = 2 - \sin(t)$$

Approaches to visualization:
A) Rotate picture.
B) Enclose curve in box (example 2).
C) Draw curve on surface (example 1).

Examples:
1) helix: r(t) = < cos(t), sin(t), t >
2) twisted cubic: r(t) = < t, t^2, t^3 >


z
C
$P(f(t), g(t), h(t))$
0
$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
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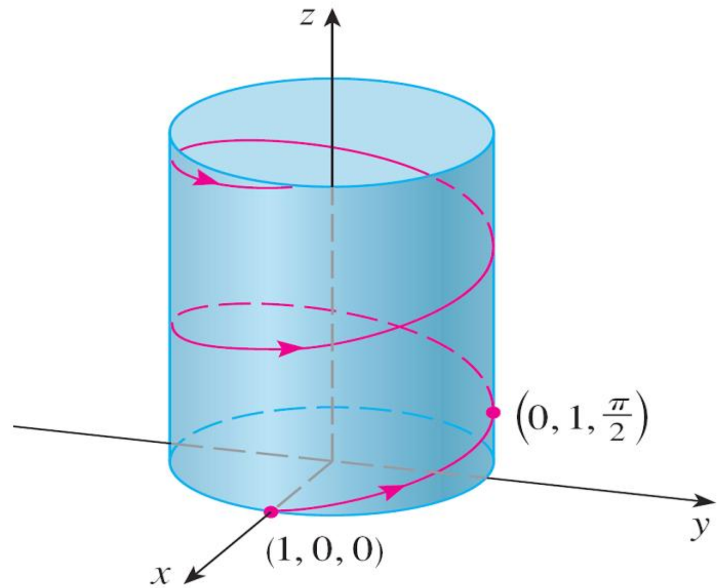
z
$\left(0, 1, \frac{\pi}{2}\right)$
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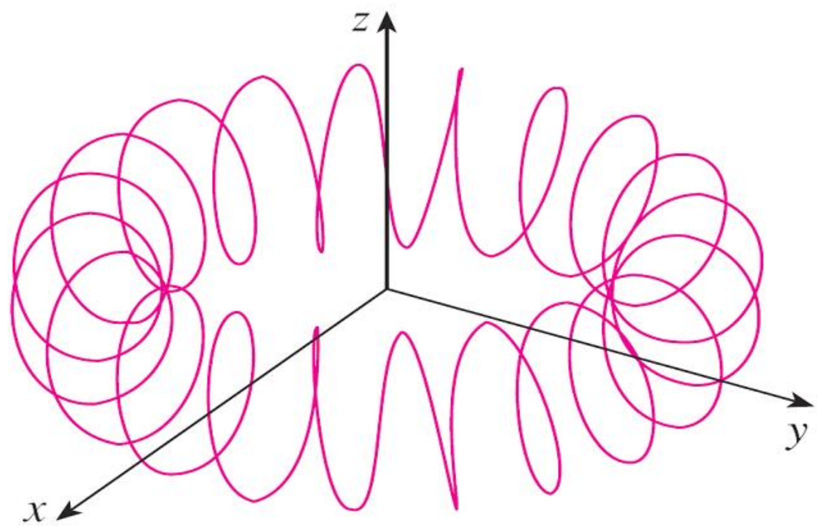
(1, 0, 0)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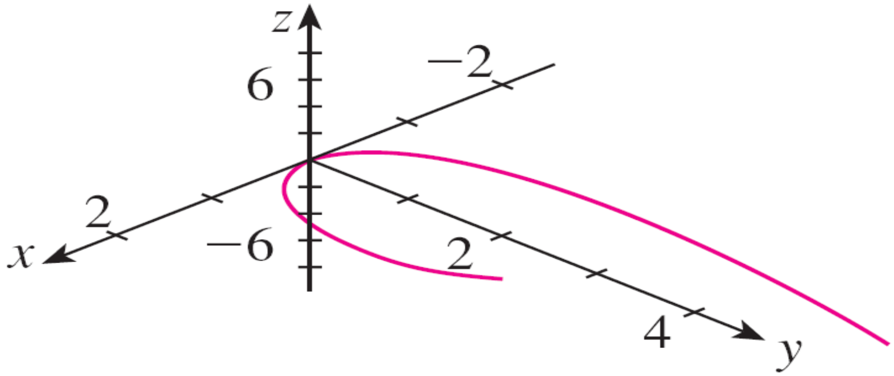
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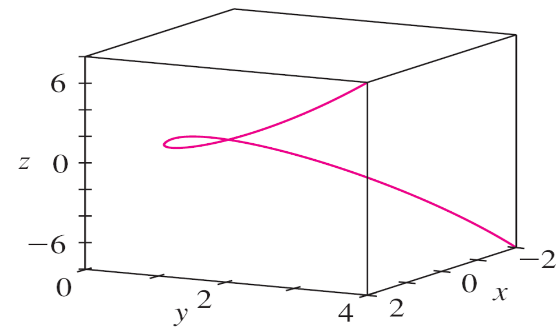
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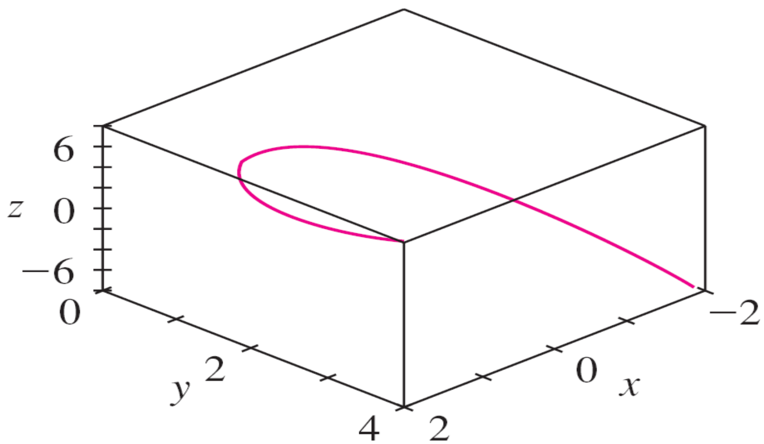
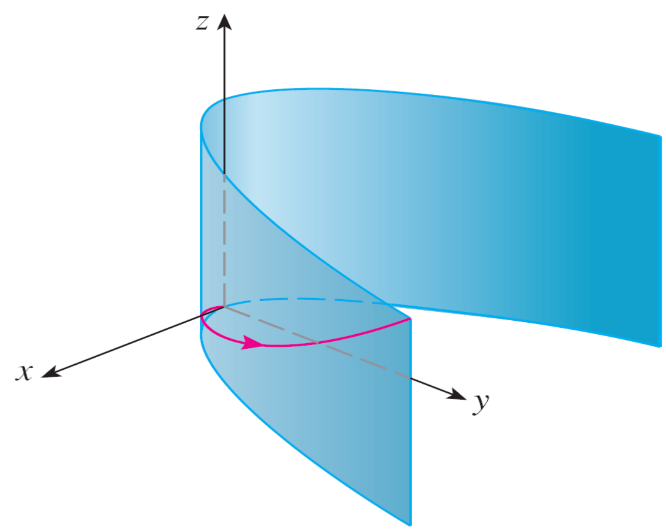
0 x
−2


6
z 0
−6
0
y 2
4
2
0 x
−2


z
x
y