New Results on Deterministic Pricing of Financial Derivatives*

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Monte Carlo simulation is widely used to price complex financial instruments. Recent theoretical results and extensive computer testing indicate that deterministic methods may be far superior in speed and confidence. Simulations using the Sobol or Faure points are examples of deterministic methods. For the sake of brevity, we refer to a deterministic method using the name of the sequence of points which the method uses, e.g., Sobol method.

In this paper we test the generalized Faure sequence due to Tezuka [**T95**]. We also test a modified Sobol method; this includes further improvements from those in [**PT95**]. We compare these two low discrepancy deterministic methods with basic Monte Carlo.

We summarize our conclusions regarding the valuation of a Collateralized Mortgage Obligation which we divide into three groups. Similar results hold for other financial instruments such as asian options.

I. Deterministic and Monte Carlo Methods.

Deterministic methods beat Monte Carlo by a wide margin. In particular,

- Both the generalized Faure and modified Sobol methods converge significantly faster than Monte Carlo.
- The generalized Faure method always converges at least as fast as the modified Sobol method and frequently faster.
- The Monte Carlo method is sensitive to the initial seed.

II. Small Number of Sample Points.

Deterministic methods outperform Monte Carlo for a small number of sample points. In particular,

- Deterministic methods attain small error with a small number of points.

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- For the hardest CMO tranche, generalized Faure achieves accuracy 10^{-2} with 170 points, while modified Sobol uses 600 points. On the other hand, the Monte Carlo method requires 2700 points for the same accuracy.
- Monte Carlo tends to *waste* points due to clustering, which severely compromises its performance when the sample size is small.

III. Speedup.
The advantage of deterministic methods over Monte Carlo is further amplified as the sample size and the accuracy demands grow. In particular,

- Deterministic methods are 20 to 50 times faster than Monte Carlo even with moderate sample sizes (2000 deterministic points or more).
- When high accuracy is desired, deterministic methods can be 1000 times faster than Monte Carlo.

Valuing financial derivatives may be formulated via paths or as a high dimensional integral. For simplicity we will restrict ourselves to the integral formulation and, without loss of generality (see Paskov [**P96**]), as an integral over the unit cube in d dimensions.

For most finance problems the integral cannot be analytically computed; we have to settle for a numerical approximation. The basic Monte Carlo method obtains this approximation by computing the arithmetic mean of the integrand evaluated at randomly chosen points. More precisely, only pseudo-random points can be generated on a digital computer and these are used in lieu of random points. There are sophisticated variations of this basic method; whenever we refer to Monte Carlo in this paper we will always mean the basic version.

If pseudo-random points from a flat distribution are plotted in two dimensions (see Figure 1) there are regions where no sample points occur and regions where the points are more concentrated. This is clearly undesirable. Random point samples are *wasted* due to clustering. Indeed, Monte Carlo simulations with very small sample sizes cannot be trusted. It would be desirable to place our sample points as uniformly as possible, which is the idea behind low discrepancy sequences. Discrepancy is a measure of deviation from uniformity; hence low discrepancy points are desirable. Figure 2 shows a plot of certain low discrepancy points in two dimensions.

A low discrepancy method approximates the integral by computing the arithmetic mean of the integrand evaluated at low discrepancy points. Low discrepancy sequences have been extensively studied; see Paskov [**P96**] for the formal definition of discrepancy and an extensive bibliography. In contrast to the Monte Carlo method which uses random (or pseudo-random) points, low discrepancy methods use deterministic points. These methods are sometimes said to be quasi-random.

In 1992 the conventional wisdom was that although theory suggested that low discrepancy methods were sometimes superior to Monte Carlo, this theoretical advantage was not seen for high dimensional problems. Traub and a Ph.D. student, Spassimir Paskov, decided to compare the efficacy of low discrepancy and Monte Carlo methods on the valuation of financial derivatives. They used a Collateralized Mortgage Obligation (Fannie Mae REMIC Trust 1989-23) provided by Goldman Sachs with ten tranches requiring the evaluation of ten 360-dimensional integrals. The values of the tranches depend on the interest rate and prepayment models. Details can be found in Paskov [**P96**].

Paskov and Traub used a particular low discrepancy sequence due to Sobol. They made major improvements in the Sobol points and showed that the improved Sobol method consistently outperformed Monte Carlo; see Paskov and Traub [**PT95**] and Paskov [**P96**] for details.

Software Construction and testing of low discrepancy deterministic methods for pricing financial derivatives was begun at Columbia University in the Fall of 1992. Preliminary results were shared with a number of New York City financial houses in the Fall of 1993 and the Spring of 1994. The first published announcement was a January 1994 article in Scientific American; Traub and Woźniakowski [**TW94**]. See Paskov and Traub [**PT95**] for a more detailed history.

In September 1995 IBM announced a product called the *Deterministic Simulation Blaster* (see also [**IBM95**]) which uses a low discrepancy deterministic method. The company claimed a very large improvement over Monte Carlo. However, the method for chosing the sample points and the methodology for calculating the speedup have not been revealed by the time of our writing this article (March 1996). IBM has repeatedly acknowledged that the use of low discrepancy methods to price financial derivatives was pioneered at Columbia University.

We have built a software system called *FINDER* for computing high dimensional integrals. *FINDER* has modules for generating generalized Faure and modified Sobol points. We emphasize that major improvements in generalized Faure points and Sobol points, which cannot be found in the published literature, have been put in *FINDER*. Indeed, a number of financial institutions have informed us that they could not replicate our results using, for example, the Sobol point generator found in Press et al. [**Pr92**]. As further improvements in low discrepancy methods are discovered they will be added to our software. *FINDER* may be obtained from Columbia University.

We used *FINDER* to price the CMO and to compare low discrepancy methods with Monte Carlo. Deterministic methods and Monte Carlo compute the arithmetic mean of the integrand evaluated at a number of points. Thus the difference in performance depends on the number of points that each method uses for the same accuracy. We observe the least number of points that a method requires in order to achieve and *maintain* a relative error below a certain level, e.g., 10^{-2}. The speedup of one method relative to another is the ratio of the least number of points required by the first method divided by the least number of points required by the second method so that both methods *maintain* the same level of accuracy. This definition of speedup is new. We study the convergence and the error of a method throughout a simulation. We feel that this has advantages over speedup calculations based only on error values at the end of a simulation. *Note that our definition of speedup is a more rigorous requirement than only computing the confidence level of Monte Carlo.*

For fixed accuracy, extensive testing has shown that different tranches require a different number of points. We emphasize that deterministic methods beat Monte Carlo for every tranche. We report results using the residual tranche of the CMO as reference point since it is the most difficult to price. The residual tranche depends on all the 360 monthly interest rates. If this tranche can be priced using a certain number of samples with a given accuracy, the same number of samples will yield at least the same accuracy for the rest of the tranches.

Since pricing models for complicated derivatives are subject to uncertainty, financial houses are often content with relative errors of one part in a hundred. Furthermore, if they wish to price a book of instruments it is critical to use a small number of samples. Deterministic methods achieve a relative error of one part in a hundred using a small number of points. In figures 3 and 4 we see that 170 generalized Faure points, 600 Sobol points, and 2700 Monte Carlo points are sufficient for a relative error equal to 10^{-2}. Thus, a very small number of generalized Faure points yields an accurate price 16 times faster than Monte Carlo.

A further reduction of the error by a factor of 20 (equal to $10^{-3}/2$) requires about 16000 generalized Faure points, while it may require up to 800000 random points and the speedup is up to 50. In general, samples using as few as 2000 generalized Faure points can price the CMO 20 to 50 times faster than Monte Carlo.

We discuss the convergence rates. For n generalized Faure points, $n \leq 10^4$, the error is proportional to $n^{-0.82}$. This error estimate is very conservative since frequently a much higher convergence rate is attained. This is very superior to the $n^{-0.5}$ expected Monte

Carlo error.

Monte Carlo exhibits a great sensitivity on the seed of the pseudo-random number generator. Unless we are dealing with the result of a fairly long simulation we cannot have much confidence. Very long simulations, needed for high accuracy in the Monte Carlo estimate, yield a deterministic method speedup on the order of 1000.

Generalized Faure points are better than Sobol points in the sense that they usually achieve the same accuracy 2.5 to 6.5 times faster. Another important advantage of the generalized Faure points is that they can be easily produced for very high dimensional problems. It is much more complicated to obtain the improved Sobol points that we have been using, in very high dimensions.

We summarize our conclusions. Among the deterministic methods we have tested, the one based on generalized Faure points is the method of choice. Generalized Faure points can be produced efficiently, at a cost similar to that required for random points, and a small number of points suffices to price the CMO. In contrast to some other deterministic sequences generalized Faure points can be easily produced in very high dimensions. *FINDER* contains features that further improve the quality of the approximation obtained by the generalized Faure method without any additional computational overhead. Finally, preliminary, but very encouraging results, indicate that generalized Faure points can efficiently price financial derivatives modeled in more than 1500 dimensions.

In closing, we indicate our plans for future work.

(1) Make further improvements in *FINDER*.
(2) Compare the performance of low discrepancy and Monte Carlo methods on other financial derivatives.
(3) Investigate the of low discrepancy methods for risk management.
(4) Design new low discrepancy methods which are especially good for financial computations.

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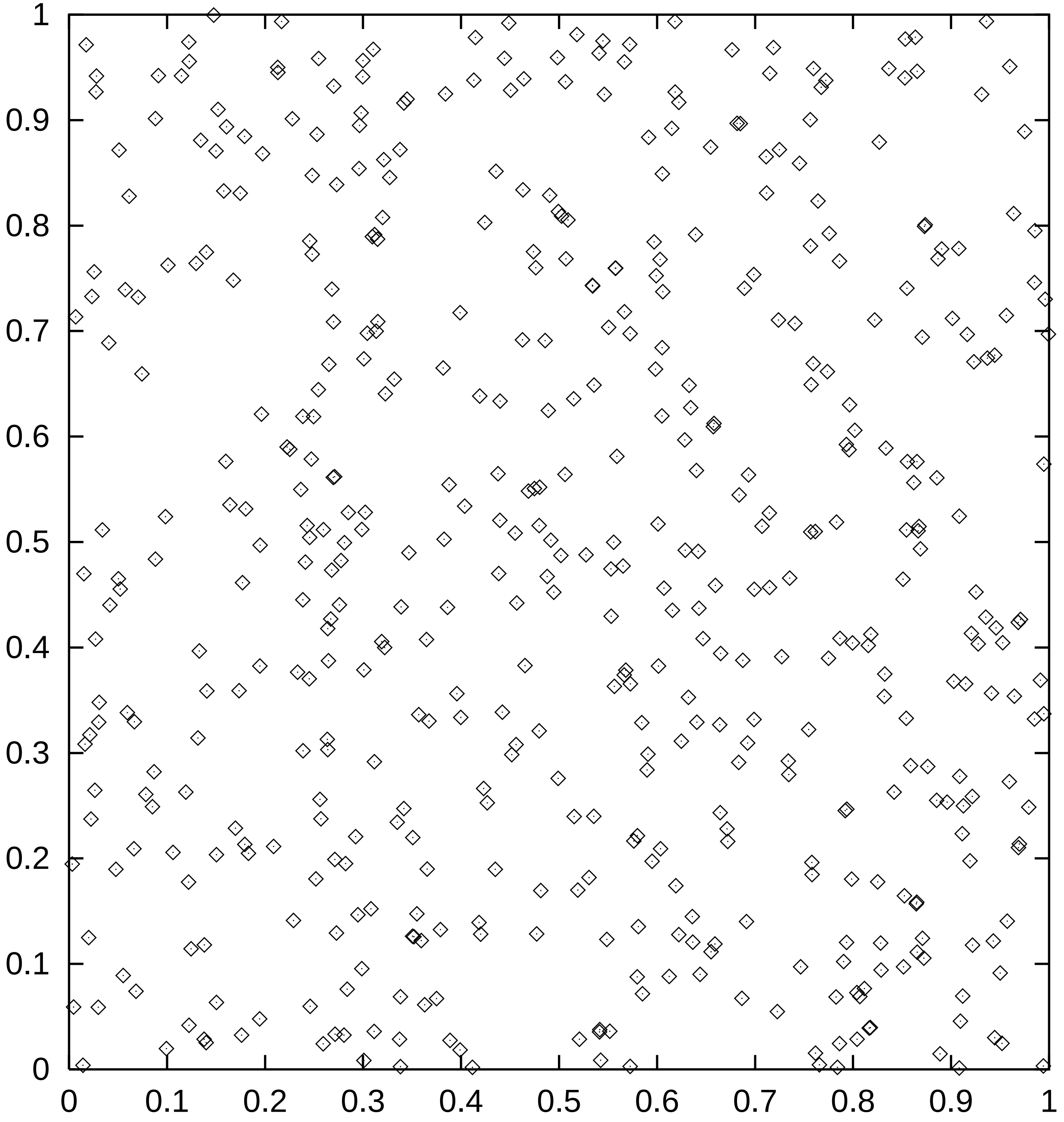

Figure 1: 512 pseudo-random points

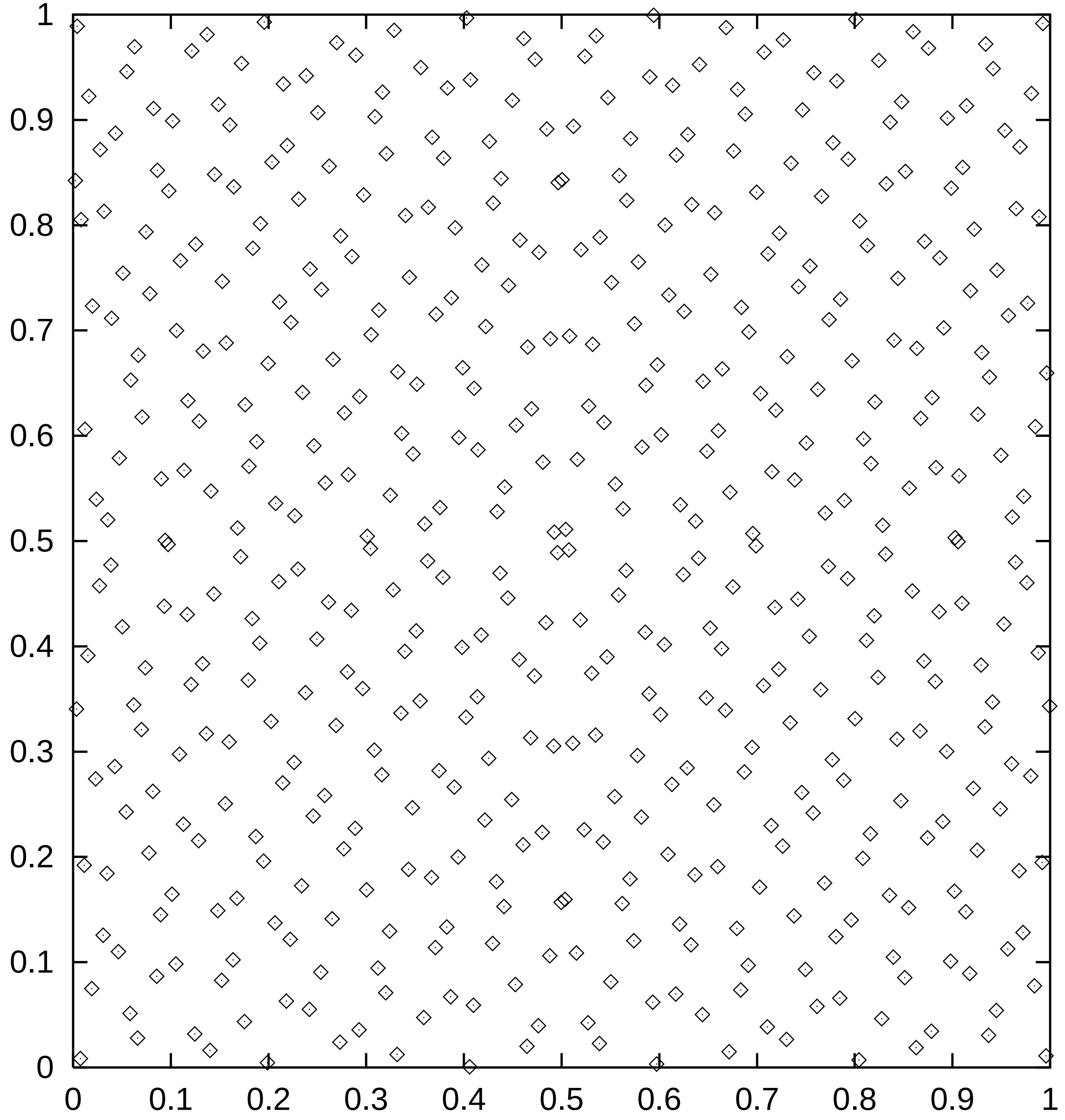

Figure 2: 512 low discrepancy points

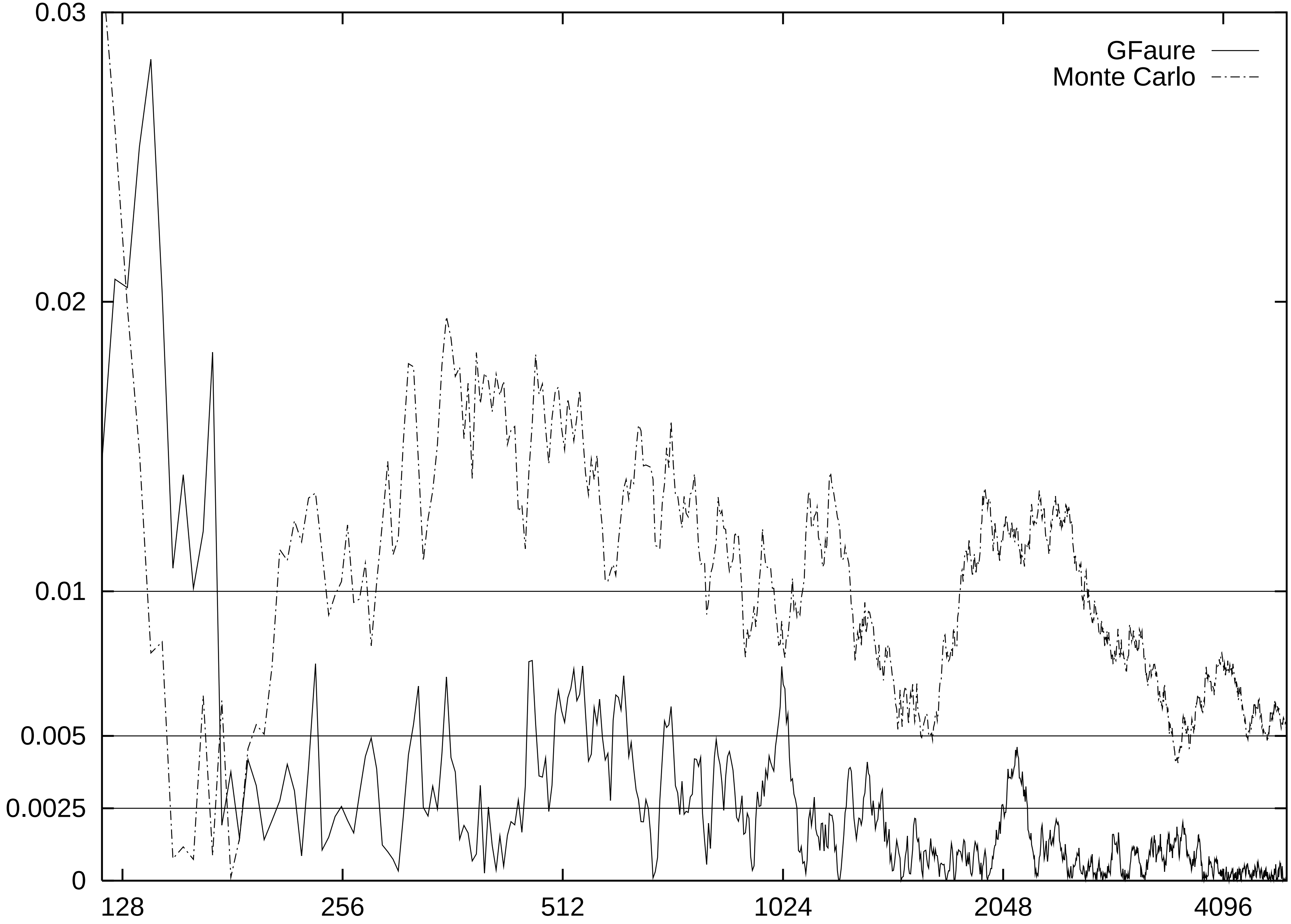

Figure 3: Generalized Faure and Monte Carlo errors

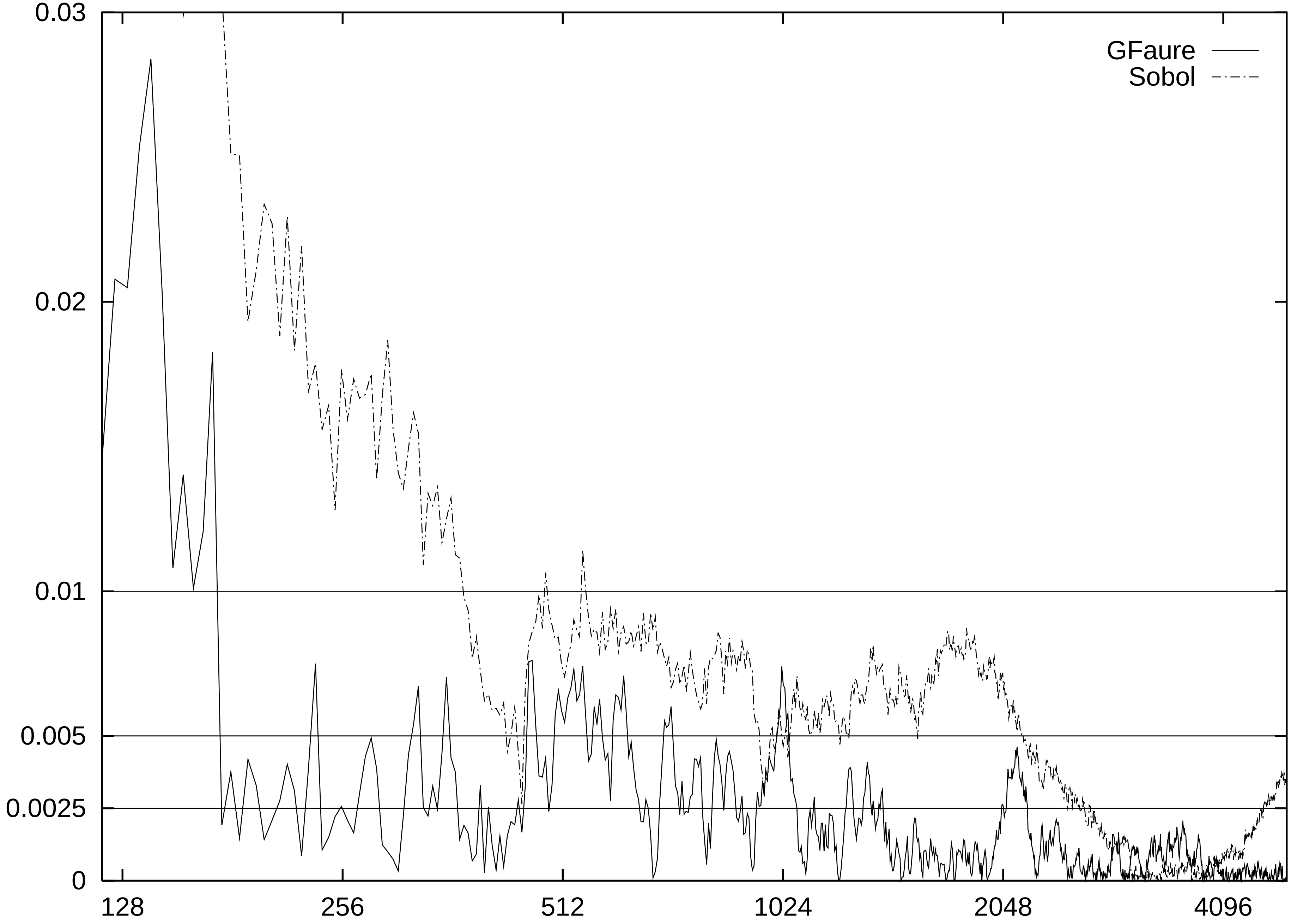

Figure 4: Generalized Faure and Sobol errors