Incentives in Computer Science (COMS 4995-6): Exercise Set #8

Due by Noon on Wednesday, March 25, 2020

Instructions:

(1) You can work individually or in a pair. If you work in a pair, the two of you should submit a single write-up.

(2) Submission instructions: We are using Gradescope for the homework submissions. Go to www.gradescope.com to either login or create a new account. Use the course code MKRKK6 to register for COMS 4995-6. Only one person needs to submit the assignment. When submitting, please remember to add your partner's name (if any) in Gradescope.

(3) Please type your solutions if possible. We encourage you to use the LaTeX template provided on the course home page.

(4) Write convincingly but not excessively. You should be able to fit all of your solutions into 2–3 pages, if not less.

(5) Except where otherwise noted, you may refer to the course lecture notes and the specific supplementary readings listed on the course Web page *only*.

(6) You can discuss the exercises verbally at a high level with other groups. And of course, you are encouraged to contact the course staff (via Piazza or office hours) for additional help.

(7) If you discuss solution approaches with anyone outside of your group, you must list their names on the front page of your write-up.

(8) Refer to the course Web site for the late day policy.

Exercise 43

Consider a two-player game in which Alice chooses either action A or action B, and Bob chooses either action C or action D. Recall that an outcome (X, Y) of the game is a *Nash equilibrium* if each player is playing a best response to the other. Equivalently, neither player can increase his or her payoff by unilaterally switching strategies.[1] Give an example of a game (with two players with two actions each) that does not have any Nash equilibria.[2]

Exercise 44

Now suppose that a game (with two players with two actions each) is played $N \geq 2$ times (where both players know N). The final payoff to a player is defined as the sum of the payoffs earned in the N "stage games." A *strategy* is now a (deterministic) function that takes as input the history-so-far (i.e., the actions taken by both players in the first i stages) and returns an action to play in the next stage. A Nash equilibrium is now defined as a strategy pair (one for Alice, one for Bob) such that each player's strategy is a best response to that of the other (i.e., no other strategy nets larger total payoff against the other's strategy). Prove that if (X, Y) is a Nash equilibrium in the stage game, then the following is a Nash equilibrium in the repeated game: Alice always plays X, and Bob always plays Y.

[1] For simplicity, we're disallowing randomized strategies. What we're calling a Nash equilibrium is usually called a *pure-strategy* Nash equilibrium, to emphasize that each player deterministically chooses an action.

[2] If randomized strategies are allowed, then every game (with any finite number of players and strategies) has at least one Nash equilibrium. (This is what Nash proved, back in 1950.)

Exercise 45

Show by example that a repeated game can have a Nash equilibrium in which the actions chosen by the players in the first stage do not constitute a Nash equilibrium of the stage game. (Two players, two actions each, and $N = 2$ suffices.)

Exercise 46

Recall the payoff matrix from lecture for the Prisoner's Dilemma:

	Cooperate	Defect
Cooperate	$2, 2$	$-1, 3$
Defect	$3, -1$	$0, 0$

Recall the Tit-for-Tat strategy: at stage 1, cooperate; at stage i, do whatever the other player did in stage $i - 1$. Prove that the Tit-for-Tat strategy never wins a head-to-head match: no matter what strategy Bob uses, if Alice uses Tit-for-Tat, then Alice's total payoff is at most that of Bob's.

Exercise 47

Consider an AS graph that is connected (i.e., every AS can reach the destination d via a sequence of direct physical connections). Consider the following procedure that attempts to compute a stable routing:

1. Initialize $H = \{d\}$ and $P_d = \emptyset$. (Here d is the destination AS.)

   [We will maintain the invariants that every AS v in H has a path P_v to d that lies entirely in H, and that $\cup_{v \in H} P_v$ forms a tree directed into d.]

2. For $u \notin H$, call a u-d path P *consistent with* H if it concludes with the v-d path P_v, where v is the first vertex of P that lies in H. (I.e., once the path hits H, it follows the unique path in H to d.)

3. While there is a vertex $u \notin H$ such that u's favorite path consistent with H has the form $(u, v) \oplus P_v$ for an edge (u, v) and a vertex $v \in H$:[3]

   (a) Add u to H.

   (b) Set $P_u = (u, v) \oplus P_v$.

4. If H is the entire vertex set, return the tree $\cup_{v \in H} P_v$. Otherwise, FAIL.

Prove that if this procedure does not FAIL, then it terminates with a stable routing.

Exercise 48

Prove that the procedure in the previous exercise can only FAIL if the given AS graph has a dispute wheel.

[Hint: use arguments reminiscent of those used to prove uniqueness in lecture.]

[3] $P \oplus Q$ denotes the concatenation of the paths P and Q. (The ending vertex of P should be the same as the starting vertex of Q.)