1. Satisfiability, Validity, Unsatisfiability. For each of the following formulas/sequents, specify whether the formula/sequent as valid/tautology, satisfiable (but not valid), or unsatisfiable and prove your answer.

   (a) $(P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))$.

   Solution: Tautology. Proof by creating truth table of all 8 assignments which all satisfy the formula.

   (b) $(P \vee Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P)$

   Solution: Satisfiable (but not valid). A satisfying assignment sets $P = 0, Q = 1, R = 0$, and a falsifying assignments sets $P = Q = 1$.

   (c) $\neg P \wedge \neg Q \rightarrow \neg(P \vee Q)$

   Solution: Tautology. Proof by truth table.

2. For the formulas from Exercise 1 that are valid, give a PK proof. (Recall that if A is valid, then a PK proof of A is a PK proof of the sequent $\rightarrow A$.)

3. For the formulas from Exercise 1 that are valid, give a Resolution refutation of the negation of the formula. (Note that if A is valid, then $\neg A$ is unsatisfiable. So you first have to show how to convert $\neg A$ into an equivalent CNF formula, using new variables as discussed in class and in the notes.)

4. (Completeness) Let A be a propositional formula and let Φ be a set of formulas. What happens when you run the derivational completeness algorithm to try to obtain a Φ-PK proof of A in the following scenarios. In particular you should specify the following for each scenario: (i) does the algorithm halt and output a Φ-PK proof of A? (ii) If the algorithm does not halt, what happens and why?

   (a) Φ is empty and A is a (propositional) formula that is not a tautology (so A is not valid). For example, $A = (x \vee y)$.

   Solution: The algorithm discussed in class technically speaking will not terminate since it will continue indefinitely to try another subformula in the infinite ordering. However since A is propositional, we could have made the process terminate and output a satisfying assignment by adapting the proof of propositional completeness (which does always terminate).

   (b) Φ is finite, and Φ does not logically imply A.

   Solution: The algorithm will get into an infinite loop.

5. Exercise 11 of Lecture Notes on Propositional Calculus (Prove equivalence of the 3 forms of compactness)

   Solution: Forms 1 and 3 are contrapositives of one another, so they are logically equivalent.

Suppose that A is a logical consequence of Φ. Then $\{\neg A, \Phi\}$ is unsatisfiable, so by Form 1, some finite subset of $\{\neg A, \Phi\}$ is unsatisfiable. Suppose this finite subset is $\{\neg A, \Phi'\}$. Then A is a logical consequence of Φ'. Thus Form 1 implies Form 2. Similarly, Suppose that Φ is unsatisfiable. Then the formula $A = (x \wedge \neg x)$ is a logical consequence of Φ so by Form 2, A is a logical consequence of some finite subset, Φ' of Φ. Since A is unsatisfiable, this implies that Φ' is also unsatisfiable. Thus Form 2 implies Form 1.

6. Prove that a theory Σ is consistent if and only if Σ has a model.

   Solution: A theory is consistent if not all sentences are in the theory. If Σ has a model, then for every sentence A, A is in Σ if and only if $\neg A$ is not in Σ (by the definition). Therefore there is at least one sentence not in Σ and therefore Σ is consistent. In the other direction, suppose that Σ is consistent. Then for every sentence A, at least one of A, $\neg A$ is not in Σ (for otherwise all sentences would be in Σ.) Then we can construct a term model for Σ. The universe consists of all terms, and as usual we define the value of all functions in the usual way. For predicates, $P(t_1, \ldots, t_k)$ is true if and only if it is in Σ. ‘

7. Exercises 4, page 25 of notes on Predicate Calculus

   Solution: P1 says that sx is not 0 – so no edges into 0; P1 says that successor relation has no 2-cycles; P3 says that $x + 0 = x$ and P4 says that $x + sy = s(x + y)$, so plus and successor are commutative. The right side says that $x + y$ always equals $y + x$.

   I think a model would be a copy of the natural numbers where plus and successor are as usual, and then an additional copy of the natural numbers. $x + y$ is defined as follows. If x, y are both from first copy then add and the resulting element is $z = x + y$ in the first copy. Similarly if x, y are in second copy then add and result is in second copy. But if one of x, y is in one copy and the other is in the other copy, then we always add and the result is in the second copy.

8. Exercise 6, page 25 of notes on Predicate Calculus

   Let M be a structure and Φ be the set of all sentences A satisfied by M. Show that if A is a logical consequence of Φ, then $A \in \Phi$.

   Solution: Since A is a logical consequence of Φ, this means that any structure M satisfying everything in Φ also satisfies A. Since everything in Φ is satisfied by M, it follows that M also satifies A and thus A is in Φ.

9. Exercises 7, page 25 of notes on Predicate Calculus

   Solution: Give a sentence involving just equality that is satisfied by a structure if and only if the universe has exactly three elements.

   $\exists x, y, z \forall w (x \neq y) \wedge (x \neq z) \wedge (y \neq z) \wedge (w = x \vee w = y \vee w = z)$.

10. Exercise 10, Herbrand, Equality, Compactness, p.50

    Let A be a first order sentence over the language with one binary predicate symbol R and equality. Suppose that for each $n \geq 3$, A has a model consisting of a directed cycle with n nodes, where R represents the edge relation of a directed graph. Prove that A has a model M whose universe includes an infinite path. (a set of distinct elements $v_0, v_1, \ldots$ such that $R^M(v_i, v_{i+1})$ holds for all $i \geq 0$.)

Solution: Add infinitely many constant symbols $c_1, c_2, \ldots$ and add to A the (infinite) sentences expressing $c_i \neq c_j$ for all $i \neq j$, and let $R(c_i, c_j)$ hold whenever $i < j$. Let A' be the resulting collection of sentences. We know that every finite subset of A' has a model. To see this, consider some finite subset A'' of A'. Suppose that it mentions at most n elements c_i. Consider a model for A of size $n' > n$, consisting of a directed cycle with n' nodes, we can associate the elements c_i that are mentioned in A'' with the elements of the model in such a way that all sentences in A'' are satisfied. Now by compactness A' also has a model, and it must include an infinite path since A' contains an infinite collection of sentences that forces an infinite path.

11. Exercise 13, Herbrand, Equality, Compactness, p.51 Use Exercise 11 above to show that $Th(s)$ (the theory of successor) is not finitely axiomatizable.

    Solution: It is not hard to show that for each $i \geq 4$, that (Si) is not a logical consequence of the previous (Sj)'s. To see this, we construct a model for (S1), ..., (Si) consisting of a copy of the natural numbers under the usual successor operation, plus an additional i elements arranged in a directed cycle. (An edge from i to j means that j is the successor of i.) Thus by Exercise 11 it follows that $Th(s)$ is not finitely axiomatizable.

12. Exercise 15, Herbrand, Equality, Compactness, p.53

    Let $\mathcal{L}$ be a language which includes an infinite list $c_1, c_2, \ldots$ of constant symbols. Let Γ be the set of sentences $c_i \neq c_j$ for all $i < j$. Let A be a sentence such that A is a logical consequence of Γ. Prove that A has a model with a finite universe.

    Solution: Since A is a logical consequence of Γ, by the completeness theorem, there is some finite subset Γ' of sentences in Γ, such that Γ' logically implies A. Let j be the largest index mentioned in the sentences of Γ'. Consider the model consisting of j elements, all distinct where equality is true equality and the relations and function symbols in A are interpreted arbitrarily. Since our model satisfies Γ' and since Γ' logically implies A, t follows that our model also satisifes A.