RESOLUTION (RES)

Whereas PK is a propositional proof system (proves tautologies), RES is a propositional refutation system (proves unsatisfiability).

Defn A literal is an atom \( P \) or its negation \( \overline{P} \).

A clause \( C = (P \lor \overline{Q} \lor \overline{R} \lor \overline{S}) \) is a disjunction of literals.

A conjunctive normal form (CNF) formula

\[ f = C_1 \land C_2 \land \ldots \land C_m \]

is a conjunction of clauses.

3SAT Theorem Let \( f \) be any propositional formula.

Then there exists a CNF formula \( f' \) such that

1. \( |f'| = 100 \cdot |f| \), and
2. \( f' \iff f \).

\[ f' : (A_1 \iff P \land Q) \land (A_2 \iff A_1 \lor \neg P) \land (A_3 \iff R \lor \neg S) \land (A_4 \iff A_2 \land A_3) \land (A_4) \]

\[ = (\neg A_1 \lor P) \land (\neg A_1 \lor Q) \land (\neg P \lor \overline{Q} \lor \overline{A_1}) \land (\neg A_2 \lor A_1 \lor \neg P) \land (\neg A_2 \lor \overline{A_1} \lor (P \lor A_2)) \land (\neg A_3 \lor R \lor \neg S) \land (\neg R \lor \overline{A_3} \lor (S \lor A_3)) \land (\neg A_4 \lor A_2 \land A_3) \land (\neg A_4 \lor A_3 \land (A_2 \lor \overline{A_3} \lor A_4)) \land A_4 \]
Resolution Rule  Let \( C_1 = (l \lor A) \), \( C_2 = (\bar{l} \lor B) \)

where \( l \) is a literal, \( A, B \) clauses.

Then \( C_1, C_2 \) derives \( C_3 = (A \lor B) \).

A resolution refutation of an unsatisfiable CNF formula \( f = C_1 \land C_2 \land \ldots \land C_m \) is a sequence of clauses \( S_1, S_2, \ldots, S_q \) such that the final clause \( S_q = \emptyset \) (the empty clause), and all other clauses \( S_i \), \( i < q \) are either a clause in \( f \), or follow from two previous clauses by the Resolution Rule.

Example  \( f = (P \lor Q) \land (\bar{Q} \lor R) \land (\bar{P} \lor S) \land (\bar{P} \lor \bar{S}) \lor (R) \)

\[
\begin{align*}
S_1 &= (P \lor Q) \\
S_2 &= (\bar{Q} \lor R) \\
S_3 &= (\bar{P} \lor S) \\
S_4 &= (\bar{P} \lor \bar{S}) \\
S_5 &= (R) \\
S_6 &= (P \lor R) \\
S_7 &= (\bar{P}) \\
S_8 &= \emptyset
\end{align*}
\]
Defn
Let $f$ be an unsatisfiable formula, and let $f'$ be the equivalent unsat formula in CNF given by the 3SAT theorem.
Then a Resolution refutation of $f$ is a resolution refutation of $f'$.

Let $g$ be a tautology. Let $f = \neg g$, and let $f'$ be the equivalent 3CNF for $f$ given by the 3SAT formula. Then a Resolution proof for $g$ is a Res refutation of $f'$.

**RES SOUNDNESS THEOREM**
Let $f$ be a CNF formula.
If $f$ has a RES refutation, then $f$ is unsatisfiable.

Let $f = c_1 \land \ldots \land c_m$, and let $P = C_1, C_2, \ldots, C_m, D_1, \ldots, D_i = \emptyset$ be a RES refutation of $f$. Prove by induction on $i$ that if $\alpha$ is a satisfying assignment to $C_1, \ldots, C_m$, then $\alpha$ is a satisfying assignment to $C_1, \ldots, C_m, D_1, \ldots, D_i$. When $i = 0$, trivial.

For $D_{i+1}$, assume $D_{i+1}$ derived from $2$ previous clauses $(x \vee E) (x \vee F) \Rightarrow D_{i+1} = (E \lor F)$. Then $\alpha$ satisfies $D_{i+1}$. But $D_i$ is unsatisfiable, thus $\alpha$ satisfies $D_{i+1}$, so $f$. 

**RES Completeness Theorem**

Let \( f \) be an unsatisfiable CNF formula. Then \( f \) has a RES refutation.

**Defn** A decision tree for \( f(x_1, x_n) \) is a rooted tree, depth \( \leq n \):

- nodes of \( T \) labelled with variables \( x_i \)
- If a node labelled \( x_i \), its two out edges labelled by literals \( x_i \) and \( \overline{x_i} \)
- A path labelling corresponds to a partial truth assignment
- A node is a leaf iff \( f^T = 0 \) or \( 1 \), where \( 0 \) is the path labelling to \( n \).

**Example**

\[
f = (x \lor \overline{z})(z \lor \overline{y})(y \lor \overline{x})(\overline{y})
\]

Claim \( f \) is unsatisfiable iff for any decision tree \( T \) for \( f \), all leaves of \( T \) are labelled 0.
Let $T$ be a decision tree for $f$, $f$ an unsatisfiable CNF formula.

Then $T$ can be converted into a RES refutation of a generalization of $f$, $f'$.

If $l_1...l_k$ labels path to $n$,
\[
\text{Clause}(n) = (\overline{l_1} \lor \overline{l_2} \lor ... \lor \overline{l_k})
\]

**Example** (from previous page)

![Decision Tree Diagram]

RES refutation of
\[
\begin{align*}
\bar{f}' &= (y \lor \neg z \lor x)(y \lor \neg z \lor \bar{x})(y \lor \bar{z})(\bar{y}) \\
f &= (x \lor \neg z)(z \lor \bar{x})(y \lor \bar{z})(\bar{y})
\end{align*}
\]

**Definition:** $f'$ is a generalization of $f$ iff for all clauses $C_i \in f'$, $\exists C_i \in f$ such that $C_i \subseteq C'$.

**Claim:** $f'$ is a generalization of $f$. 
Proof of Claim

By construction of $T$ for $f$, for each leaf node $l$ of $T$, there exists a clause $C_{X|f}$ that is falsified by the path to $l$.

By construction of $T^*$, for each leaf node $l$ of $T$, the clause $C^*$ associated with $l$ is defined to be the maximum clause that is falsified by the path to $l$.

Thus, for each leaf $l$, $C_{X|f} \leq C^*$. 
Proof of Resolution Completeness

1. Let \( f(x_1, \ldots, x_n) \) be an unsatisfiable CNF formula.

2. Create a decision tree \( T \) for \( f \). By claim, all leaves of \( T \) are labelled by 0.

3. Convert \( T \) to a RES refutation \( N^* \) of \( \phi' = \bigwedge C_i \), where \( \phi' \) is a generalization of \( f \).
   That is, for all \( C_i \in \phi' \), there exists a clause \( C_i \in f \) such that \( C_i \leq C_i' \).

4. (Exercise) Show that for any \( \phi' \), \( f \) where \( \phi' \) is a generalization of \( f \), a RES refutation of \( \phi' \) can be converted into a RES refutation of \( f \).

DLL Resolution Procedure

- Order \( x_1 \) to \( x_n \)
- Create two querying clauses in order unless there is a unit clause.