RESOLUTION (RES)

Whereas PK is a propositional proof system (proves tautologies)
RES is a propositional refutation system (proves unsatisfiability)

<u>Def'n</u> A <u>literal</u> is an atom P or its negation $\bar{P}$

A <u>clause</u> $C = (P \vee \neg Q \vee R \vee S)$ is a disjunction of literals.

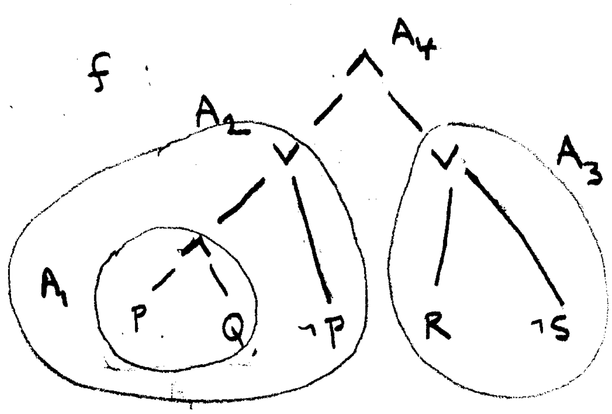
A <u>conjunctive Normal form</u> (CNF) formula

$f = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ is a conjunction of clauses.

3SAT
<u>Theorem</u> Let f be any propositional formula.
Then there exists a CNF formula f' such that
(1) $|f'| \leq 100 \cdot |f|$, and (2) $f' \Leftrightarrow f$.

$|f|$ = number of $\wedge, \vee$ connectives in f

f

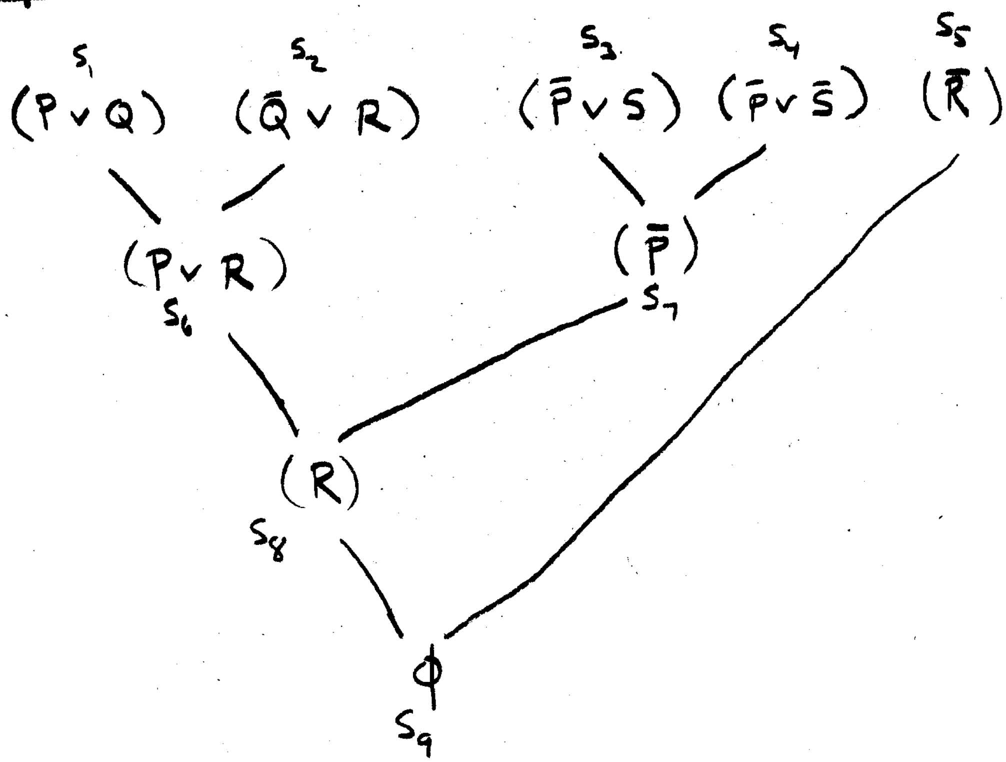


$$
f' : \begin{array}{l} (A_1 \leftrightarrow P \wedge Q) \wedge \\ (A_2 \leftrightarrow A_1 \vee \neg P) \wedge \\ (A_3 \leftrightarrow R \vee \neg S) \wedge \\ (A_4 \leftrightarrow A_2 \wedge A_3) \wedge \\ (A_4) \end{array}
\equiv
\overbrace{\begin{array}{l} (\neg A \vee P) \wedge (\neg A \vee Q) \wedge (\bar{P} \vee \bar{Q} \vee A_1) \wedge \\ (\bar{A}_2 \vee A_1 \vee \neg P) \wedge (\bar{A}_1 \vee A_2) \wedge (P \vee A_2) \wedge \\ (\bar{A}_3 \vee R \vee \neg S) \wedge (\neg R \vee A_3) \wedge (S \vee A_3) \wedge \\ (\bar{A}_4 \vee A_2) \wedge (\bar{A}_4 \vee A_3) \wedge (\bar{A}_2 \vee \bar{A}_3 \vee A_4) \wedge \\ A_4 \end{array}}^{f'}
$$

<u>Resolution Rule</u> Let $C_1 = (\ell \vee A)$, $C_2 = (\bar{\ell} \vee B)$ where ℓ is a literal, A, B clauses.
Then C_1, C_2 derives $C_3 = (A \vee B)$.

A resolution refutation of an unsatisfiable CNF formula $f = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ is a sequence of clauses $S_1, S_2, \ldots, S_q$ such that the final clause $S_q = \phi$ (the empty clause); and all other clauses S_i, $i<q$ are either a clause in f, or follow from two previous clauses by the Resolution Rule.

<u>Example</u> $f = (P \vee Q) \wedge (\bar{Q} \vee R) \wedge (\bar{P} \vee S) \wedge (\bar{P} \vee \bar{S}) \vee (\bar{R})$



Defn

Let f be an unsatisfiable formula, and let f' be the equivalent UNSAT formula in CNF given by the 3SAT theorem.
Then a Resolution refutation of f is a resolution refutation of f'.

Let g be a tautology. ~~Let~~ $f = \neg g$, and let f' be the equivalent 3CNF for f given by the 3SAT formula. Then a Resolution proof for g is a Res refutation of f'.

RES SOUNDNESS THEOREM Let f be a CNF formula. If f has a RES refutation, then f is unsatisfiable.

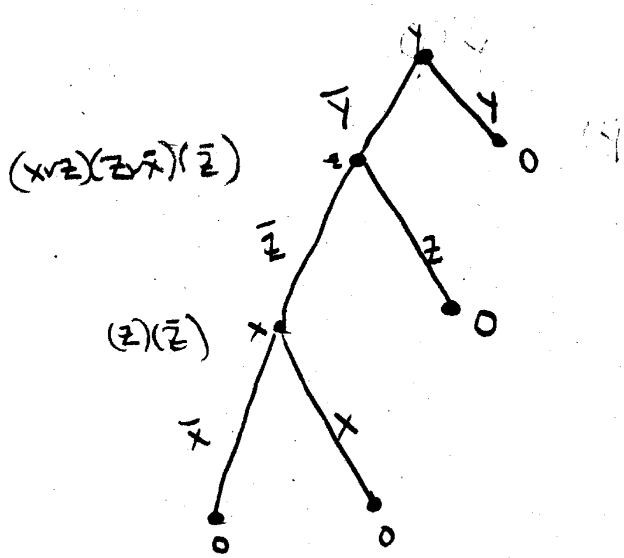
Let $f = C_1 \wedge \ldots \wedge C_m$, and let $P = C_1, C_2, \ldots, C_m, D_1, \ldots, D_q = \phi$ be a RES refutation of f. Prove by induction on i that if α is a satisfying assignment to $C_1 \wedge \ldots \wedge C_m$ then α is a satisfying assignment to $C_1 \wedge \ldots \wedge C_m \wedge D_1 \wedge \ldots \wedge D_i$. When $i = 0$, trivial.
For D_{i+1}, assume D_{i+1} derived from 2 previous clauses $(x \vee E)$ $(\bar{x} \vee F)$ $\Rightarrow D_{i+1} = (E \vee F)$. Then α satisfies D_{i+1}. But D_q is unsatisfiable, thus so is f.

RES COMPLETENESS THEOREM Let f be an unsatisfiable CNF formula. Then f has a RES refutation.

Def'n A *decision tree* for $f(x_1 \ldots x_n)$ is a rooted tree, depth $\leq n$.

- nodes of T labelled with variables x_i
- If a nonleaf node labelled x_i, its two outedges labelled by literals x_i and $\bar{x}_i$.
- A path labelling corresponds to a partial truth assignment
- A node n is a leaf iff $f\restriction_\sigma = 0$ or 1, where σ is the path labelling to n.

Example $f = (x \vee z)(\bar{z} \vee \bar{x})(y \vee \bar{z})(\bar{y})$



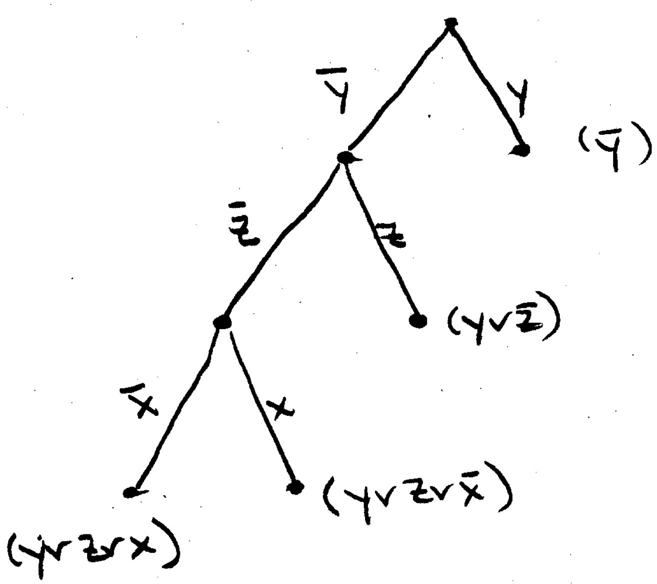
Claim f is unsatisfiable iff for any decision tree T for f, all leaves of T are labelled 0.

Let T be a decision tree for f, f an unsatisfiable CNF formula.

Then T can be converted into a RES refutation of a generalization of f, f'.

- If $l_1 \ldots l_k$ labels path to n,
  $\text{Clause}(n) = (\bar{l}_1 \vee \bar{l}_2 \vee \ldots \vee \bar{l}_k)$.

<u>Example</u> (from previous page)



RES refutation of

$$f' = (y \vee z \vee x)(y \vee z \vee \bar{x})(y \vee \bar{z})(\bar{y})$$
$$f = (x \vee z)(z \vee \bar{x})(y \vee \bar{z})(\bar{y})$$

<u>Def'n</u> f' is a generalization of f iff for all clauses $C_i' \in f'$, $\exists C_i \in f$ such that $C_i \subseteq C_i'$.

<u>Claim</u> f' is a generalization of f.

Proof of Claim

By construction of T for f, for each leaf node ℓ of T, there exists a clause $C_\ell \in f$ that is falsified by the path to ℓ.

By construction of T^*, for each leaf node ℓ of T, the clause C^*_ℓ associated with ℓ is defined to be the maximum clause that is falsified by the path to ℓ.

Thus, for each leaf ℓ, $C_\ell \subseteq C^*_\ell$.

Proof of Resolution Completeness

1. Let $f(x_1 .. x_n)$ be an unsatisfiable CNF formula

2. create a decision tree T for f. By claim, all leaves of T are labelled by 0.

3. Convert T to a RES refutation Π^* of $f' = C_1' \wedge \ldots \wedge C_m'$, where f' is a generalization of f. That is, for all $C_i' \in f'$, there exists a clause $C_i \in f$ such that $C_i \subseteq C_i'$.

4. (Exercise) Show that for any f', f where f' is a generalization of f, a RES refutation of f' can be converted into a RES refutation of f.

ordered
DLL Resolution Procedure

Order vars $x_1 - x_n$
Create tree, querying vars in order
unless there is a unit clause.