FIRST-ORDER RESOLUTION (See Buss, Intro to Proof Theory)

We now consider another proof system for predicate logic that is also SOUND + COMPLETE and very popular in theorem proving.

FO RES will be an extension of (propositional) RES.

How to use Resolution as a FIRST ORDER theorem prover?

- Convert an arbitrary formula into a generalized CNF formula. Terms "stand in" for quantifiers (Normal Form)
- generalize Resolution to apply to CNF formulas, but now containing terms (Herbrand Thm / unification)

Conversion to Normal Form

① Convert A to prenex Normal form, $A' = Q_1 x_1 \ldots Q_n x_n B$

Q_i is $\exists$ or $\forall$ quantifier-free CNF

- use unique variable names in quantification
- move negation inwards $\neg \forall x B \Leftrightarrow \exists x \neg B$
  $\neg \exists x B \Leftrightarrow \forall x \neg B$
- move quantifiers to left $(\forall x B \wedge C) \Leftrightarrow \forall x (B \wedge C)$
  $(\forall x B \vee C) \Leftrightarrow \forall x (B \vee C)$
  $(\exists x B \wedge C) \Leftrightarrow \exists x (B \wedge C)$
  $(\exists x B \vee C) \Leftrightarrow \exists x (B \vee C)$

Convert B to CNF by distributivity

② Skolemize to eliminate all existentially quantified variables. } functional form

Remove each $\exists y$ in prefix
Replace y in formula by $f_y(x_1 \ldots x_k)$
where $x_1 \ldots x_k$ are the universally quantified vars preceding $\exists y$; f_y a new function symbol

③ Apply a variable renaming so that for any 2 clauses C_1, C_2, the variables in C_1 are disjoint from variables in C_2.
Remove universal quantification (assumed implicitly)

Example

$$\forall x\,((A(x) \vee \neg \exists y\, B(x,y)) \wedge \exists z\, C(z,x))$$

(1) $\forall x\, \exists z\, \forall y\, [(A(x) \vee \neg B(x,y)) \wedge (C(z,x))]$

prenex and CNF

(2) $\forall x\, \forall y\, [\underbrace{(A(x) \vee \neg B(x,y))} \wedge \underbrace{(C(fx,\, x))}]$

Clauses

(3) $(A(x) \wedge \neg B(x,y)) \wedge (C(fz, z))$

Theorem Let A be a FO formula, A' the Normal form of A. Then A is satisfiable iff A' is satisfiable.

Same holds for Φ a set of formulas.

Unification (see Buss chapter 1, 2.6)

The job of unification is to take two atomic formula p, q and return the most general substitution that makes p and q syntactically identical.

t is a term containing function + constant symbols and variables.

A <u>substitution</u> σ is a partial map from variables to terms.

Let $A_1 \ldots A_k$ be atomic formulas. A <u>unifier</u> for $\{A_1 \ldots A_k\}$ is a substitution σ such that

$$A_1\sigma = A_2\sigma = \ldots = A_k\sigma$$

syntactic equivalence

A unifier σ for $A_1 \ldots A_n$ is a <u>most general unifier</u> if $\forall$ unifiers τ for $A_1 \ldots A_n$, $\exists \rho$ s.t. $\tau = \sigma\rho$
(apply σ then ρ)

σ <u>most general</u> if $\forall$ unifiers τ for same set, $\exists \rho$ s.t. $\tau = \sigma\rho$.

<u>Claim</u> up to var renaming, a most general unifier is unique

UNIFICATION ALGORITHM

Input: Atomic formulas $A(\bar{p})$, $A(\bar{q})$

Output: the most general unifier, σ, of $\bar{p}$ and $\bar{q}$ if one exists

At stage s, algorithm maintains

E_s : set of equations to unify

σ_s : substitution

(1) Initially $E_0 = \{ p_1 = q_1, p_2 = q_2, \dots, p_n = q_n \}$

$\sigma_0 = \{ x := x, \; y := y, \dots \}$ (identity)

(2) Stage s+1 (given E_s, σ_s)

- If $E_s = \phi$, halt and return σ_s
- If E_s contains an equation

  (*) $F(t_1 \dots t_i) = F(t_1', \dots, t_i')$, then

  function symbol

  $\sigma_{s+1} = \sigma_s$

  E_{s+1} : remove (*) from E_s, and add $t_1 = t_1'$

  $t_2 = t_2'$

  $t_i = t_i'$

(3) If E_s contains an equation

(**) $x = t$

variable term

If $t = x$, $E_{s+1} = E_s$ but with (**) removed

If $t \neq x$ but contains x, halt + return FAIL

otherwise

E_{s+1} is the set of equations $s[x/t] = s[x/t]$

where $s = s'$ is in E_s

$\sigma_{s+1} = \sigma_s[x/t]$

substitution mapping x to t

$x := t$

Example $\text{Unify}[P(x, F(x, A)), P(H(y), z)]$

1. $E_0 = P(x, F(x, A)) = P(H(y), z))$

   $\sigma = \text{identity} \quad x := x \quad y := y \quad z := z$

2. $E_1 = \{x = H(y), F(x, A) = z\}$

   $\sigma_1 = \sigma_0$

3. $E_2 = \{F(H(y), A) = z\}$

   $\sigma_2 = x := H(y) \quad y := y \quad z := z$

4. $E_3 = \{\}$

   $\sigma_3 = x := H(y) \quad y := y \quad z := F(H(y), A)$

Generalized Resolution

Let A be a FO formula, and let A' be the normal form of A.

$A' = \forall\, B_1 \wedge \dots \wedge B_m$

"clauses" or disjunctions of atomic formulas

A Resolution refutation of A' is a sequence of atomic formulas $L_1, L_2, \dots, L_q$ where each L_i is either a clause of A or follows from two previous L_k's by the generalized Resolution rule:

Let B be a clause containing atomic formulas $P(\bar{s}_1), \dots, P(\bar{s}_k)$ (and possibly other atomic formulas)

Let C be a clause containing atomic formulas $\neg P(\bar{t}_1), \dots, \neg P(\bar{t}_\ell)$

Let σ be a most general unifier for $\{P(\bar{s}_1), \dots, P(\bar{s}_k), P(\bar{t}_1), \dots, P(\bar{t}_\ell)\}$

Derive clause $D = (B\sigma \setminus P(\bar{s}_1)\sigma) \cup (C\sigma \setminus \neg P(\bar{t}_1)\sigma)$

resolve away all $P(\bar{s}_i)\sigma$, $P(\bar{t}_j)\sigma$

FO completeness Theorem for Resolution

A is unsatisfiable iff there is a First order Resolution refutation of A' (the Normal form of A).

Completeness Proof

<u>Defn</u> A <u>ground literal</u> is an atomic formula or the negation of an atomic formula in which NO variables occur.

A <u>ground clause</u> is the disjunction of a set of ground literals.

<u>FACT</u> By completeness of Propositional Resolution, a set of ground clauses is unsatisfiable iff it has a (ground) Resolution refutation

Now Let A' be a set of unsatisfiable FO clauses. (A' are the clauses in the normal form of A, and may not consist of ground literals.)

By <u>Herbrand's Theorem</u> there is a set of substitutions $\sigma_1 \ldots \sigma_r$ s.t. each $A'\sigma_i$ is a set of ground clauses and so that $\bigcup_i A'\sigma_i = \Pi$ is propositionally unsatisfiable.
Thus, there is a ground resolution refutation of Π.

<u>Show</u> Any ground RES refutation can be <u>lifted</u> to a FO RES refutation of A'.

Lifting Ground Resolution to FO Resolution

Ground Literal: no variables
Ground Clause: a set of ground literals

By Completeness of Propositional Resolution, a set of ground clauses is unsat. iff it has a ground Res refutation

- Let f be an unsat. FO formula, f' the normal form of f. By Herbrand's Theorem, there is a set of substitutions $\sigma_1, \sigma_2 \dots \sigma_r$ such that $\Pi = \bigcup_i f'\sigma_i$ is an unsatisfiable set of ground clauses
- By prop. Res completeness, Π has a Res refutation
- show any ground Res refutation of Π can be lifted to a FO RES ref of f'.

Show If $C_1 C_2 \dots C_n = \phi$ is a Res refutation of Π, $\exists D_1 D_2 \dots D_m = \phi$ which forms a FO RES refutation of f' and there are substitutions $\sigma_1 \sigma_2 \dots \sigma_n$ such that $D_i\sigma_i = C_i$

Pf by induction on i

① $C_i \in \Pi$. Then $C_i = D_i\sigma_i$ for some $D_i \in f'$, and some σ_i

② $\dfrac{C_j \quad C_k}{C_i}$ on literal $P(\bar{r})$.

Define E_j to be the subset of D_j which mapped to $P(\bar{r})$ by σ_j. Likewise for E_k.
then unify E_j E_k and resolve to get D_i