More on term Models, Herbrand's Theorem and FIRST ORDER RESOLUTION

Recall in our proof of completeness of LK we showed:

Let $\mathcal{A}$ be any set of sentences.
If $\mathcal{A}$ is not valid, then there is a <u>term model</u> $\mathcal{M}$ such that all $A \in \mathcal{A}$ are falsified by $\mathcal{M}$.

A <u>term model</u> for $\mathcal{L}$: universe of $\mathcal{M}$ is all possible $\mathcal{L}$-terms.

Defn A term t is a ground term if t contains no variables

Definition Let $A = \forall x_1 \forall x_2 \ldots \forall x_k B$, $k \geq 0$, B quantifier-free.
A ground instance of A is of the form $B(t_1/x_1, t_2/x_2, \ldots, t_k/x_k)$ where $t_1 \ldots t_k$ are ground terms

Fact Every ground instance of A is a logical consequence of A.
$\therefore$ if a set Φ_0 of ground instances of A is unsatisfiable then A is unsatisfiable

Defn (Propositional Satisfiability/UNSAT of $\forall B$ sentences)

A (propositional) truth assignment τ maps each $\mathcal{L}$-atomic formula to $\{0,1\}$

We can extend τ to all quantifier-free formulas via inductive defn of propositionally satisfiable

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Ex. $\Phi = \{(\forall x P(x) \vee \forall y Q(y)), \neg \forall x P(x), \neg \forall y Q(y)\}$

$\mathcal{L} = \{f, g, c; P, Q\}$

$$\left.\begin{array}{l} P(g(c)) \\ Q(g(c)) \\ Q(f(c)) \end{array}\right\} \begin{array}{l} \mapsto 0 \\ \mapsto 0 \\ \mapsto 1 \end{array}$$

$$\underbrace{\underbrace{P(g(c))}_{0} \vee \underbrace{Q(g(c))}_{0}}_{0}$$

As a corollary, we get:

Herbrand's Theorem Let $\mathcal{L}$ be a first order language with at least one constant symbol (= zero-ary function symbol). Let Φ be a set of $\forall$ $\mathcal{L}$-sentences. Then Φ is unsatisfiable iff some finite set of $\mathcal{L}$-ground instances of sentences in $\hat{\Phi}$ is propositionally unsatisfiable

sentence of form $\forall A$

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Proof Let $\hat{\Phi}$ be a set of ground instances of Φ

① $\hat{\Phi}$ prop unsat $\longrightarrow$ Φ prop. unsat (by FACT)

② show: If every finite subset of ground instances of Φ is SAT, then Φ is SAT

Proof ② show: If every finite subset of ground instances of Φ is SAT, then Φ is SAT

Let Φ_0 be all ground instances of Φ

- By propositional compactness, if every finite subset of Φ_0 is propositionally satisfiable, then Φ_0 is propositionally satisfiable.

Proof ② show: If every finite subset of ground instances of ϕ is SAT, then Φ is SAT

Let Φ_0 be all ground instances of ϕ

- By propositional compactness, if every finite subset of Φ_0 is propositionally satisfiable, then Φ_0 is propositionally satisfiable.
- Let τ be a propositional truth assignment that satisfies Φ_0
- We use τ to construct a term model that satisfies ϕ:

$$M = \text{all ground } \mathcal{L}\text{-terms}$$

$$c \text{ a } 0\text{-ary function symbol}: \quad c^M = \hat{c}$$

$$f^M(\hat{t}_1, \dots, \hat{t}_n) = \widehat{f(t_1 \dots t_n)}$$

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- c a 0-ary function symbol: $c^M = \hat{c}$
- $f^M(\hat{t}_1, \ldots, \hat{t}_n) = \widehat{f(t_1 \ldots t_n)}$
- $P^M(\hat{t}_1, \ldots, \hat{t}_n) = 1$ iff $P(t_1 \ldots t_n)^\tau = 1$

- By induction $\mathcal{M} \models B$ iff $B^\tau = 1$ (for all quantifier free sentences B)

$\therefore$ $\mathcal{M}$ satisfies ϕ (defn of satisfies for all sentences in ϕ)

Herbrand's Theorem Let $\mathcal{L}$ be a First order language with at least one constant symbol (= zero-ary function symbol). Let Φ be a set of $\mathcal{L}$-sentences. Then Φ is unsatisfiable iff some finite set of $\mathcal{L}$-ground instances of sentences in Φ is propositionally unsatisfiable.

Proof (2) show: If every finite subset of ground instances of Φ is SAT, then Φ is SAT.

Let Φ_0 be all ground instances of Φ

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Models of $\mathcal{L}_A$

Recall $\mathcal{L}_A = \{0, s, +, \cdot\ ;\ =\}$ Language of arithmetic

the **standard model** for $\mathcal{L}_A$: $\mathbb{N}$

$M = \mathbb{N}$, $0, s, +, \cdot$ have usual meanings

Th(A) or TA: the set of all sentences of $\mathcal{L}_A$ that are true in $\mathbb{N}$

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A nonstandard model of $\mathcal{L}_A$: any model of $\mathcal{L}_A$ that is not isomorphic to the standard model $\mathbb{N}$

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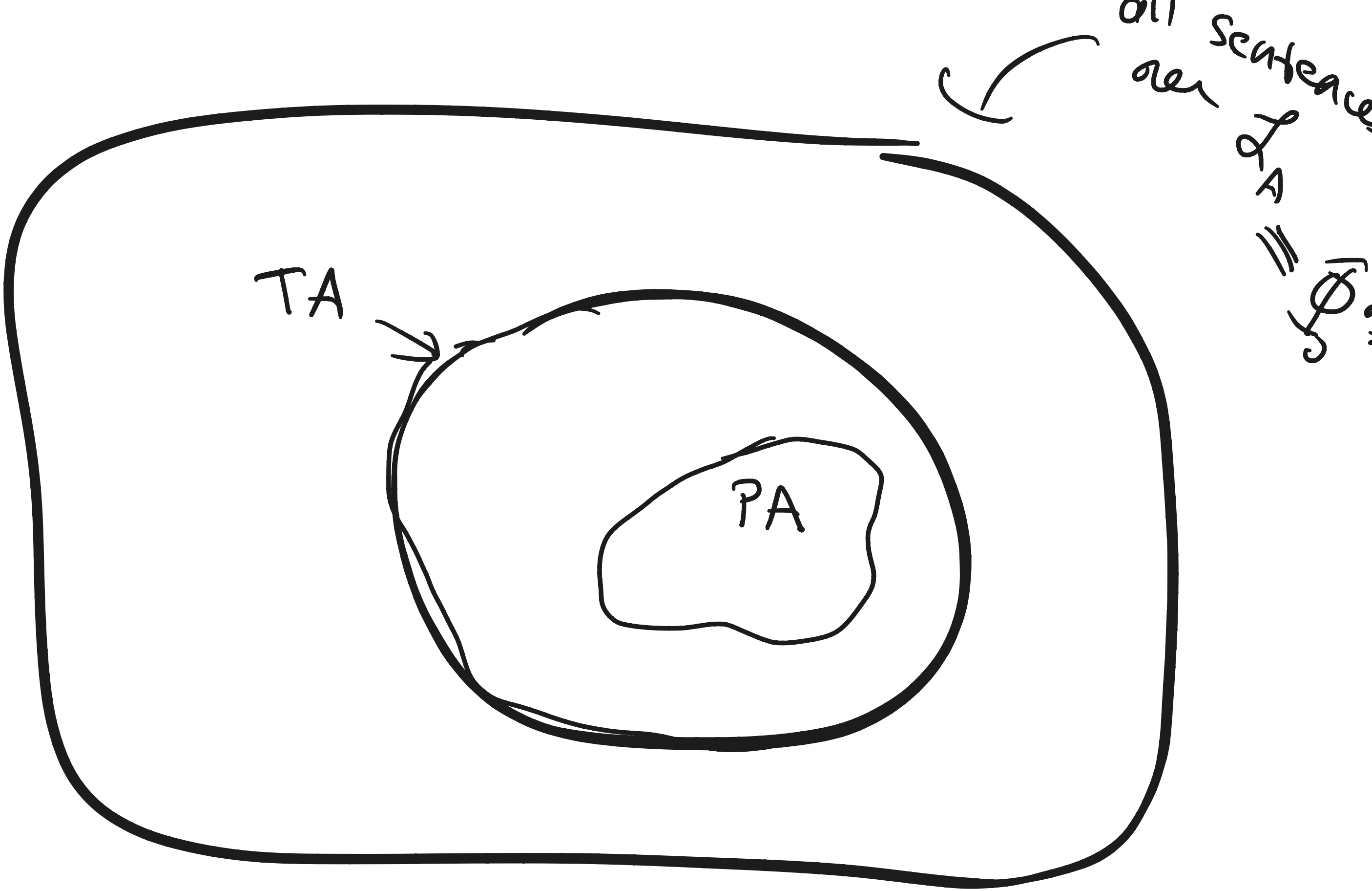
or TA: the set of all sentences of $\mathcal{L}_A$ that are true in $\mathbb{N}$

Defn A set Φ of sentences is decidable if there is an algorithm (that always halts) that given a sentence B, outputs 1 if B is in Φ and otherwise outputs 0


all sentences
over $\mathcal{L}_A$
$\equiv \Phi_{\mathcal{L}_A}$
TA
PA

We will soon see that TA is **not** decidable.

On the other hand, restricted systems of TA <u>are</u> decidable ($\mathcal{L}_s$, $\mathcal{L}_+$)

Theories

Note: JN Lecture notes this is not defined until p. 75 but it is important enough that we introduce it now.

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Definition A theory (over $\mathcal{L}$) is a set Σ of sentences closed under logical consequence. ($\Sigma \models A$ then $A \in \Sigma$)

We can specify a theory by a finite or countable set of sentences Ψ -- the theory corresponding to Ψ is $\{A \mid \Psi \models A\}$

Notation Σ a theory $\Sigma \vdash A$ means $A \in \Sigma$

Definition For a language $\mathcal{L}$, $\Phi_0^{\mathcal{L}}$ is the set of all sentences over $\mathcal{L}$

Theories

Definition Σ is consistent if and only if $\Sigma \neq \Phi_0$

(If $\Sigma = \Phi_0$ then Σ contains A & $\neg A$

Conversely if Σ contains A & $\neg A$ then Σ contains all of Φ_0)

Theories

Definición Σ is **consistent** if and only if $\Sigma \nvdash \Phi_0$

Σ is **complete** iff Σ is consistent and for all sentences A, either $\Sigma \vdash A$ or $\Sigma \vdash \neg A$

Theories

Definición Σ is consistent if and only if $\Sigma \nvdash \Phi_0$

Σ is complete iff Σ is consistent and for all sentences A, either $\Sigma \vdash A$ or $\Sigma \vdash \neg A$

Example $\mathcal{L}_A = \{0, s, +, \cdot\,; =\}$

TA = all sentences over $\mathcal{L}_A$ that are true in $\mathbb{N}$

is consistent and complete

Theories

Definición Σ is **consistent** if and only if $\Sigma \nvdash \Phi_0$

Σ is **complete** iff Σ is consistent and for all sentences A, either $\Sigma \vdash A$ or $\Sigma \vdash \neg A$

Definición A theory Σ over $\mathcal{L}_A$ is **sound** iff $\Sigma \subseteq TA$

Subsystems of True Arithmetic

- Theory of Successor $(0, s; =)$
- Presburger Arithmetic $(0, s, +; =)$
- Peano Arithmetic $(0, s, +, \cdot; =)$

Defn $\mathcal{L}_s = \{0, s; =\}$ Language of successor

The standard model for $\mathcal{L}_s$, $\mathbb{N}_s$:

$M = \mathbb{N}$, 0 and s have usual meaning $(s(x) = x+1)$

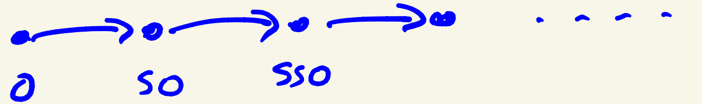
Let $Th(s)$ (theory of successor) be the set of all sentences of $\mathcal{L}_s$ that are true in $\mathbb{N}_s$

Th(S): There is a simple (infinite but countable) complete set of axioms for Th(S), Ψ_S

Ψ_S:
(S1) $\forall x\,(Sx \neq 0)$
(S2) $\forall x \forall y\,(Sx = Sy \supset x = y)$
(S3) $\forall x\,(x = 0 \vee \exists y\,(x = Sy))$
(S4) $\forall x\,(Sx \neq x)$
(S5) $\forall x\,(SSx \neq x)$
(S6) $\forall x\,(SSSx \neq x)$
(S7)
⋮

Models for Ψ_s: A model for Ψ_s is a model/structure over $\mathcal{L}_s$ that satisfies all formulas in Ψ_s

(1)



← isomorphic to $\mathbb{N}$ up to renaming

(2)

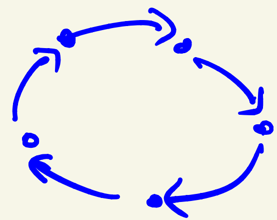
plus

$\mathbb{N}$ plus a copy of integers

③ generalizing ②, models contain one copy of $\mathbb{N}$, plus any number of copies (isomorphic to) the integers

Note without all axioms S4, S5, S6, ...
we could have additional models with loops

plus



any number of cycles

Theorem Ψ_s is complete and consistent

(proof omitted)

Therefore although Ψ_s has both the standard model $\mathbb{N}$ as well as nonstandard models, all models $\mathcal{M}$ of Ψ_s have the same set of true sentences.

<u>Theorem</u> Ψ_S is complete and consistent

(proof omitted)

Therefore although Ψ_S has both the standard model $\underline{\mathbb{N}}$ as well as nonstandard models, all models $\mathcal{M}$ of Ψ_S have the <u>same</u> set of true sentences.

We'll see later that when a set of sentences (such as $Th(S)$) has a nice (enumerable) axiomatization, then $Th(S)$ is decidable.

Defn $\mathcal{L}_+ = \{0, s, + \,;\, =\}$ Language of Presburger arithmetic

the standard model for $\mathcal{L}_+$, $\underline{\mathbb{N}_+}$:

$M = \mathbb{N}$, $0, s, +$ have usual meaning

$\underline{Th(+)}$: (theory of Presburger arithmetic, or standard model for $\mathcal{L}_+$): all sentences of $\mathcal{L}_+$ that are true in $\underline{\mathbb{N}_+}$

Presburger (1928) showed that $Th(+)$ is also characterized by a countable set of axioms (like the theory of successor)

so it is also consistent and complete

Peano Arithmetic

$\mathcal{L}_A = \{0, S, +, \cdot \, ; =\}$

- Has countable (and decidable) set of axioms
- We think it is consistent
- Has standard model $\mathbb{N}$
- Also has not tame nonstandard models

PA is a theory

BACK TO TA (TRUE ARITHMETIC)

the standard model for $\mathcal{L}_A$, $\underline{\mathbb{N}}$:

$M = \mathbb{N}$, $0, s, +, \cdot$ have usual meaning

Th(A) or TA: (Theory of True Arithmetic): set of all $\mathcal{L}_A$ sentences that are true in standard model $\underline{\mathbb{N}}$

Theorem TA has a nonstandard model

Theorem TA has a nonstandard model

Proof Let c be a constant symbol (not in $\mathcal{L}_A$)

$$\Psi = \{ c \neq 0,\ c \neq s0,\ c \neq ss0,\ c \neq sss0,\ \ldots \}$$

- every finite subset of Ψ is satisfiable
- so by compactness, $TA \cup \Psi$ has a model $\mathcal{M}$
- M is not isomorphic to $\underline{\mathbb{N}}$ (standard model) since c cannot be any element of $\mathbb{N}$

MIDTERM REVIEW

Material covered:

1. Propositional Calculus (pp 1-17 of Notes and Notes on Resolution ~~)~~ + PK)
2. Predicate Calculus (pp 18-30 of Notes)
3. Completeness (pp. 31-38 of Notes)
4. ~~Equality Axioms~~ (pp. 43-47)
   Corollaries of Completeness (48-53)

MIDTERM REVIEW

Study Tips

- Read Lecture Notes and Course Notes carefully first
- Then do/review solutions to homework questions and tutorial problems
- Then do practice questions (see handout "Midterm Study Problems")

MIDTERM REVIEW

Study Tips

- given a propositional or first order formula/sequent, produce a (RES, PK, LK) proof
- Run completeness Algorithm(s)
- Compactness: what is it? how to use it? why is it true?
- give a model for Φ; does $\Phi \models A$? is Φ valid? satisfiable? invalid/unsat.?