TODAY:

1. COMPLETENESS THEOREM

   (pages 31-38 of notes)

2. COROLLARIES: Lowenheim-Skolem, Compactness

   (pages 48-49)

3. EQUALITY (pages 43-48)

FIRST ORDER SEQUENT CALCULUS LK

Lines are again sequents

$$A_1, \dots, A_k \to B_1, \dots, B_\ell$$

where each A_i, B_j is a proper $\mathcal{L}$-formula

RULES

OLD RULES OF PK

PLUS NEW RULES FOR $\forall, \exists$

like a large AND

Large OR

New Logical Rules for $\forall$, $\exists$

$\forall$-left
$$\frac{A(t), \Gamma \to \Delta}{\forall x A(x), \Gamma \to \Delta}$$

$\forall$-Right
$$\frac{\Gamma \to \Delta, A(b)}{\Gamma \to \Delta, \forall x A(x)}$$

$\exists$-left
$$\frac{A(b), \Gamma \to \Delta}{\exists x A(x), \Gamma \to \Delta}$$

$\exists$-right
$$\frac{\Gamma \to \Delta, A(t)}{\Gamma \to \Delta, \exists x A(x)}$$

* A, t are proper
* b is a free variable Not appearing in lower sequent of rule

SOUNDNESS

Defn A first order sequent $A_1, \dots, A_k \rightarrow B_1, \dots, B_\ell$ is valid if and only if its associated formula $(A_1 \wedge \dots \wedge A_k) \supset (B_1 \vee \dots \vee B_\ell)$ is valid.

Soundness Theorem for LK Every sequent provable in LK is valid

TODAY: gödels COMPLETENESS THEOREM

Defn An LK-Φ proof is an LK-proof, but leaves are either axioms ($A \Rightarrow A$) or of the form $\Rightarrow A$ for $A \in \Phi$

goal prove that if $\Gamma \Rightarrow \Delta$ is a logical consequence of Φ, then there is an LK-Φ proof of $\Gamma \Rightarrow \Delta$ (called Derivational completeness)

Defn Let $A(a_1 .. a_n)$ be a formula with free variables $a_1 .. a_n$. Then $\forall A$ is $\forall x_1 \forall x_2 .. \forall x_n A(x_1 .. x_n)$ (called universal closure of A)

TODAY: LK COMPLETENESS

(MAIN LEMMA) Completeness Lemma

If $\Gamma \to \Delta$ is a logical consequence of a set of (possibly infinite) formulas $\forall \Phi$ then there exists a finite subset $\{C_1, \dots, C_n\}$ of Φ such that

$\forall C_1, \dots, \forall C_n, \Gamma \to \Delta$ has a (cut-free) PK proof

* We will assume = Not in Language for now

Derivational Completeness Theorem

Let Φ be a set of sequents or formulas such that the sequent $\Gamma \to \Delta$ is a logical consequence of $\forall\Phi$.
Then there is an LK-Φ proof of $\Gamma \to \Delta$.

$\Uparrow$

Proof follows from Completeness Lemma (similar to derivational completeness of PK from completeness)

Proof of LK Completeness Lemma

High Level idea (assume Φ is empty for now)

- As in PK completeness, we want to construct an LK proof in reverse.
- Start with $\Gamma \Rightarrow \Delta$ at root, and apply rules in reverse (to break up a formula into one or 2 smaller ones)

- Tricky rules are $\exists$right & $\forall$left.
  When applying one of these in reverse, Need to "guess" a term

New Logical Rules for $\forall$, $\exists$

$\forall$-left $$\frac{A(t), \Gamma \rightarrow \Delta}{\forall x A(x), \Gamma \rightarrow \Delta}$$

$\forall$-Right $$\frac{\Gamma \rightarrow \Delta, A(b)}{\Gamma \rightarrow \Delta, \forall x A(x)}$$

$\exists$-left $$\frac{A(b), \Gamma \rightarrow \Delta}{\exists x A(x), \Gamma \rightarrow \Delta}$$

$\exists$-right $$\frac{\Gamma \rightarrow \Delta, A(t)}{\Gamma \rightarrow \Delta, \exists x A(x)}$$

* A, t are proper
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Proof of LK Completeness Lemma

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- As in PK completeness, we want to construct an LK proof in reverse.
- Start with $\Gamma \Rightarrow \Delta$ at root, and apply rules in reverse (to break up a formula into one or 2 smaller ones)
- Tricky rules are $\exists$right & $\forall$left. When applying one of these in reverse, Need to "guess" a term
- Key is to systematically try all possible terms — without going down a rabbit hole.

Example of an LK proof

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{Pa \to Pa}{Pa, Qa \to Pa}}{Pa \wedge Qa \to Pa}}{Pa \wedge Qa \to \exists x Px}}{\exists x(Px \wedge Qx) \to \exists x Px}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{Qa \to Qa}{Pa, Qa \to Qa}}{Pa \wedge Qa \to Qa}}{Pa \wedge Qa \to \exists x Qx}}{\exists x(Px \wedge Qx) \to \exists x Qx}
}{
\exists x(Px \wedge Qx) \to \exists x Px \wedge \exists x Qx
}
$$

?

?

Example of an LK proof

$$Pa, Qa \rightarrow Pb$$

$$\dfrac{Pa \wedge Qa \rightarrow Pb}{\dfrac{Pa \wedge Qa \rightarrow \exists x Px}{\exists x (Px \wedge Qx) \rightarrow \exists x Px}} \qquad \dfrac{Pa \wedge Qa \rightarrow \exists x Qx}{\exists x (Px \wedge Qx) \rightarrow \exists x Qx}$$

$$\overline{\exists x (Px \wedge Qx) \rightarrow \exists x Px \wedge \exists x Qx}$$

Instead:

$$Pa, Qa \rightarrow Pb, \exists x Px$$

$$Pa \wedge Qa \rightarrow Pb, \exists x Px$$

$$\dfrac{\dfrac{Pa \wedge Qa \rightarrow \exists x Px}{\exists x (Px \wedge Qx) \rightarrow \exists x Px} \qquad \dfrac{Pa \wedge Qa \rightarrow \exists x Qx}{\exists x (Px \wedge Qx) \rightarrow \exists x Qx}}{\exists x (Px \wedge Qx) \rightarrow \exists x Px \wedge \exists x Qx}$$

Instead

Try again

$$Pa, Qa \rightarrow Pb, Pfa, \exists x Px$$

$$Pa, Qa \rightarrow Pb, \exists x Px$$

$$Pa \wedge Qa \rightarrow Pb, \exists x Px$$

$$\dfrac{\dfrac{Pa \wedge Qa \rightarrow \exists x Px}{\exists x (Px \wedge Qx) \rightarrow \exists x Px} \qquad \dfrac{Pa \wedge Qa \rightarrow \exists x Qx}{\exists x (Px \wedge Qx) \rightarrow \exists x Qx}}{\exists x (Px \wedge Qx) \rightarrow \exists x Px \wedge \exists x Qx}$$

Instead

and again

$\vdots$

and again

$Pa, Qa \to Pb, Pfa, Pfb, \exists x Px$

Try again

$$\dfrac{Pa, Qa \to Pb, \exists x Px}{Pa, Qa \to Pb, Pfa, \exists x Px}$$

$$\dfrac{Pa \wedge Qa \to Pb, \exists x Px}{Pa \wedge Qa \to \exists x Px}$$

$$\dfrac{\dfrac{Pa \wedge Qa \to \exists x Px}{\exists x (Px \wedge Qx) \to \exists x Px} \qquad \dfrac{Pa \wedge Qa \to \exists x Qx}{\exists x (Px \wedge Qx) \to \exists x Qx}}{\exists x (Px \wedge Qx) \to \exists x Px \wedge \exists x Qx}$$

There are infinitely many choices! Need a systematic way to try all

Instead

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{Pa, Qa \to Pb, Pfa, Pfb, \exists x Px}}{Pa, Qa \to Pb, Pfa, \exists x Px}}{Pa, Qa \to Pb, \exists x Px}}{\dfrac{Pa \wedge Qa \to Pb, \exists x Px}{\dfrac{Pa \wedge Qa \to \exists x Px}{\exists x(Px \wedge Qx) \to \exists x Px}}} \qquad \dfrac{Pa \wedge Qa \to \exists x Qx}{\exists x(Px \wedge Qx) \to \exists x Qx}}{\exists x(Px \wedge Qx) \to \exists x Px \wedge \exists x Qx}$$

and again

and again

Try again

There are infinitely many choices! Need a systematic way to try all and for all frontier sequents in current proof!

Completeness: Proof Search Algorithm

Enumeration of formulas + terms:

Since the number of underlying symbols of $\mathcal{L}$ is finite, there is an <u>enumeration</u> of pairs $\langle A_1, t_1 \rangle, \langle A_2, t_2 \rangle, \langle A_3, t_3 \rangle, \ldots$ such that <u>every</u> term and <u>every</u> formula in $\mathcal{L}$ occur infinitely often in the enumeration

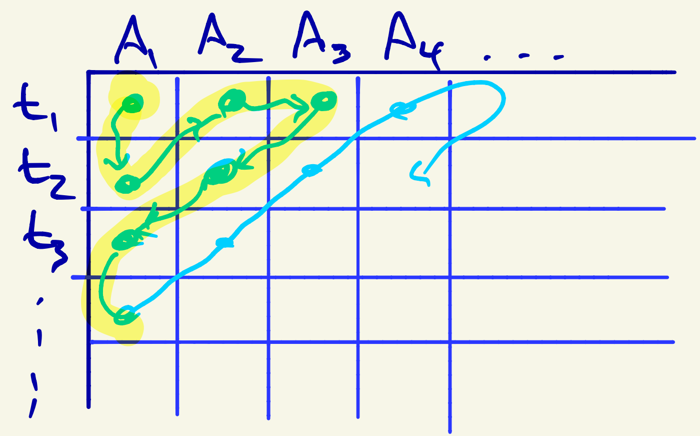
More details of enumeration ($\mathcal{L}$ finite)

Enumerate all $\mathcal{L}$-formulas $A_1, A_2, \ldots$

Enumerate " $\mathcal{L}$-terms $t_1, \ldots$

such that every formula/term occurs infinitely often

Enumerate all pairs to have same property



Completeness: Proof Search Algorithm

Start with Φ = set of sequents/formulas, $\Gamma \to \Delta$

Want an algorithm that will output an Φ-LK proof of $\Gamma \to \Delta$ whenever $\Phi \models \Gamma \to \Delta$

- Initially π is the sequent $\Gamma \to \Delta$
- At each stage, modify π by adding some $A_i \in \Phi$ to antecedent of all sequents in π, and adding onto the "frontier" or "active" sequents in π.
- Active sequent: a leaf sequent in π, not a weakening of $A \to A$
- at stage k: we will use the k^{th} pair $\langle A_k, t_k \rangle$ in the enumeration

Completeness: Proof Search Algorithm

Stage k: $\langle A, t \rangle_k$

(1) If $A_k \in \Phi$, replace $\Gamma' \Rightarrow \Delta'$ in Π by $\Gamma', A_k \Rightarrow \Delta'$

(2) If A_k atomic, skip this step. Otherwise for all leaf sequents containing A_k, break up outermost connective of A_k using the appropriate logical rule, and t_k if necessary.

Completeness: Proof Search Algorithm

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Examples:

- $A_k = \exists x B x$

$$\frac{\Gamma, B(c) \to \Delta}{\Gamma, \exists x B(x) \to \Delta}$$

c is a new variable

$$\frac{\Gamma \to \Delta, \exists x B(x), B(t_k)}{\Gamma \to \Delta, \exists x B(x)}$$

keep both here

Completeness: Proof Search Algorithm

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(1) If $A_k \in \Phi$, replace $\Gamma' \to \Delta'$ in Π by $\Gamma', A_k \to \Delta'$

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Examples:

- $A_k = \forall x\, B(x)$

$$\frac{\Gamma \to \Delta, B(c)}{\Gamma \to \Delta, \forall x\, B(x)}$$

c a new variable

$$\frac{B(t_k), \forall x\, B(x), \Gamma \to \Delta}{\Gamma, \forall x\, B(x) \to \Delta}$$

Keep both here

Exit when no more active sequents

Proof of Correctness

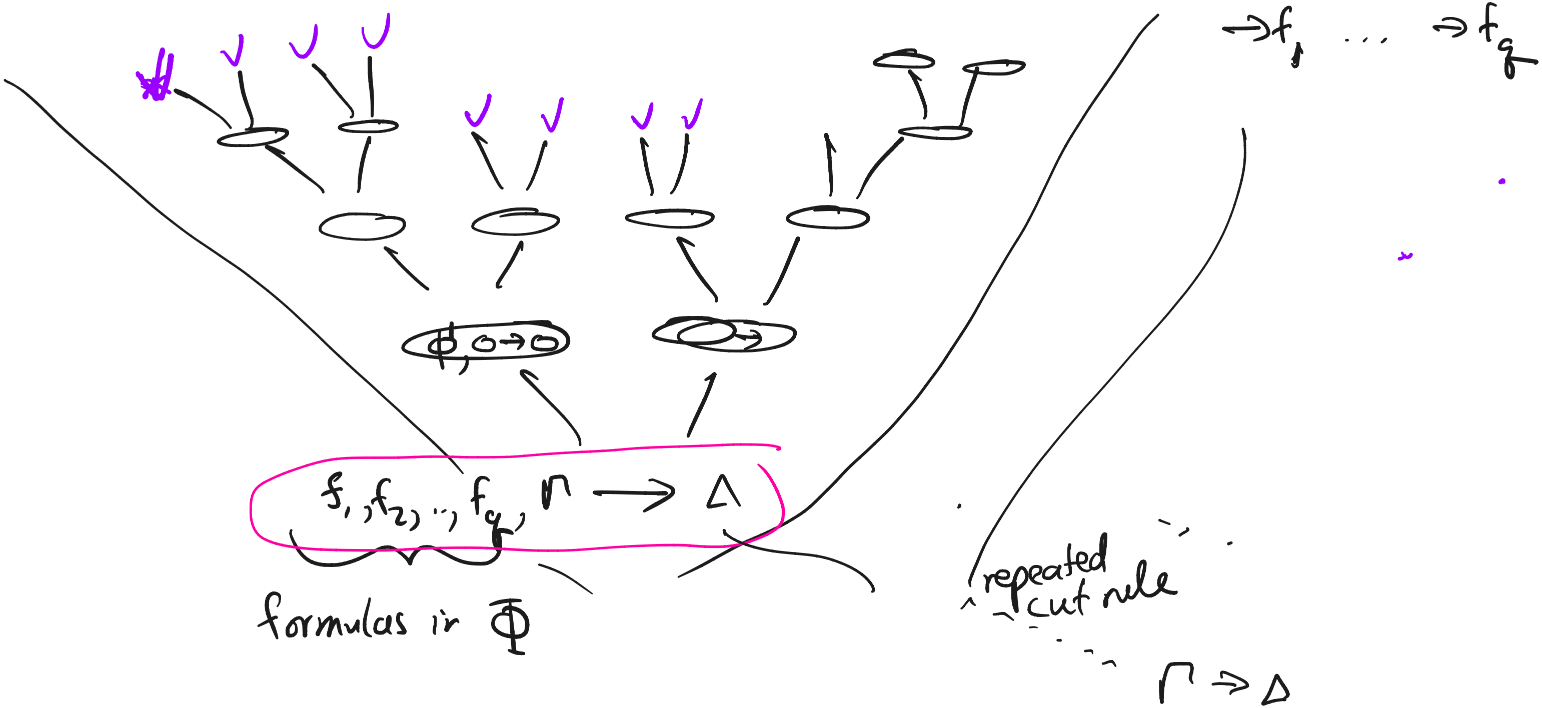
We want to show:

- If Algorithm halts, Π is an LK-Φ proof of $\Gamma \to \Delta$ ✓

- If Algorithm never halts, then $\forall \Phi \not\vdash \Gamma \to \Delta$

Show: If our algorithm halts when run on Φ, $\Gamma \to \Delta$ then it produces a Φ-LK "proof" of $\Gamma \to \Delta$

What will proof tree look like if Alg halts?



Proof of Correctness

We want to show: If Algorithm never halts, then $\Phi \not\models \Gamma \to \Delta$
(or if it halts but some leaf sequent doesn't contain some formula A on both left + rt of $\to$)

Suppose Algorithm doesn't halt and let Π be the (typically infinite) tree that results

Leaf "sequents" of Π look like $\Gamma', \underbrace{C_1, C_2, \ldots}_{\text{infinite sequence containing all of } \Phi \text{ each infinitely often}} \to \Delta'$

Find a bad path β in the tree:
If Π finite, $\exists$ some active leaf node containing only atomic formulas. Choose β to be path from root to this leaf

Proof of Correctness

We want to show: If Algorithm never halts, then $\forall \Phi \not\models \Gamma \to \Delta$

Find a bad path β in the tree:

If Π finite, $\exists$ some active leaf node containing only atomic formulas. Choose β to be path from root to this leaf

If Π infinite by König's Lemma, $\exists$ an infinite path. Let β be this path

Proof of Correctness

Properties of β

(1) β is a path starting at root

(2) all sequents in β were once active

(3) for all sequents in β, no formula occurs on both the Left and right side of sequent

(4) all atomic formulas $A \in \Phi$ in root sequent of β on LEFT, and thus occur on LEFT of all sequents in β

By (3)+(4), we know that NO atomic $A \in \Phi$ occurs on the Right of any sequent in β

Proof of Correctness (cont'd)

We will construct a "term" model $\mathfrak{M}$, + object assignment σ from β such that $\mathfrak{M} \models \Phi[\sigma]$ but $\mathfrak{M} \not\models \Gamma \Rightarrow \Delta$ (and thus our algorithm fails to halt + produce a proof <u>only</u> when $\Gamma \Rightarrow \Delta$ is not a logical consequence of Φ.)

Let M (universe) be all terms over $\mathcal{L}$

Nice thing is then our interpretation of all functions is natural: If we have a term

$$f : \text{pairs of universe elems} \rightarrow \text{universe element}$$

$$f(\overline{S1}, \overline{SSS1}) = \underbrace{\overline{fS1SSS1}}$$

Proof of correctness (cont'd)

We will construct a "term" model $\mathfrak{M}$, + object assignment σ from β such that $\mathfrak{M} \models \Phi[\sigma]$ but $\mathfrak{M} \not\models \Gamma \to \Delta$

Universe M: all $\mathcal{L}$-terms t (containing only free vars)

σ : map variable $\underline{a}$ to itself ($\sigma(a) = \bar{a}$)

$$f^{\mathfrak{M}}(\bar{r}_1 \ldots \bar{r}_k) \stackrel{d}{=} \overline{f r_1 \ldots r_k}$$

$P^{\mathfrak{M}}(r_1 \ldots r_k) \stackrel{d}{=}$ true if and only if $P r_1 \ldots r_k$ is on the LEFT of some sequent in β

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathcal{M}, \sigma$ satisfies A iff A is on the LEFT of some sequent in β, and

$\mathcal{M}, \sigma$ falsifies A iff A is on the RIGHT of some sequent in β

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathcal{M}_{,\sigma}$ satisfies A iff A is on the LEFT of some sequent in β, and

$\mathcal{M}_{,\sigma}$ falsifies A iff A is on the RIGHT of some sequent in β

Proof (induction on A)

A atomic: A cannot occur on LEFT of some sequent in β and on RIGHT of some sequent in β (since A persists up β)

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathfrak{M}, \sigma$ satisfies A iff A is on the LEFT of some sequent in β, and

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Proof (induction on A)

Induction Step Example $A = \exists x\, B(x)$ on RIGHT

high level: if A occurs in some sequent in β, then A persists upward until it becomes the active formula (at stage k, $A_k = A$) then use inductive hypothesis

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathfrak{M}, \sigma$ satisfies A iff A is on the LEFT of some sequent in β, and

$\mathfrak{M}, \sigma$ falsifies A iff A is on the RIGHT of some sequent in β

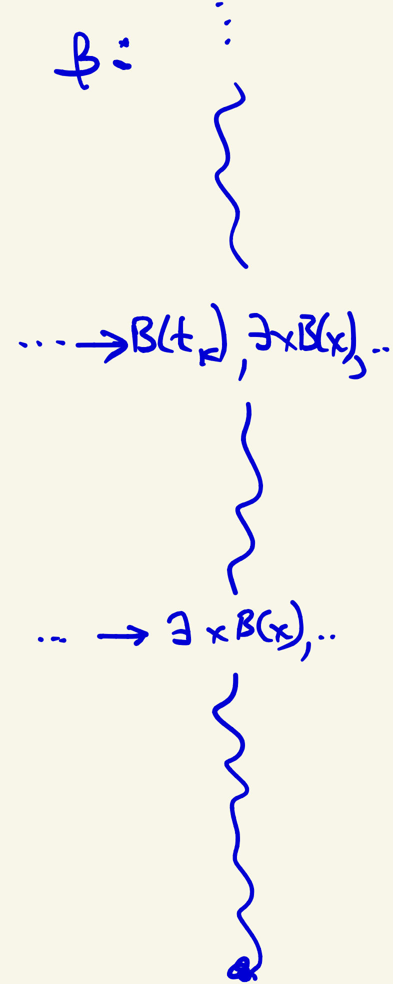
Proof (induction on A)

Induction Step $A = \exists x B(x)$ on RIGHT

By Ind hyp, M, σ falsify $B(t_j)$

Since $\exists x B(x)$ persists, we have $\forall t$ $B(t)$ on RIGHT of some sequent in β

Thus $\mathfrak{M}, \sigma$ falsify $B(t)$ for all terms t



Corollaries of Completeness

1. **Lowenheim-Skolem Theorem**. Let $\mathcal{L}$ be countable, Φ a set of sentences over $\mathcal{L}$.

   Φ satisfiable $\Rightarrow$ Φ is satisfiable in a countable universe.

Proof Follows from completeness proof. Let Φ be satisfiable

Let $A = \; \longrightarrow$ (empty sequent is unsatisfiable)

Then $\Phi \not\models A$, so proof of completeness constructs a countable model where Φ is satisfiable. $\boxtimes$

Algorithm for finding a Φ-LK proof of $\to A$

Enumerate all $\langle B, t\rangle$ — B a formula over $\mathcal{L}$, t is a term

s.t. every B, t occur infinitely often

Step i use $\langle B,t\rangle_i$:

1. If $B \in \Phi$ then add B to LHS of all sequents

2. If B not atomic, then:

   For every active sequent containing B, apply the appropriate rule in reverse.

   Use t if $B = \exists x A(x)$ & occurs on Rt

   Use t if $B = \forall x A(x)$ & occurs on left

Corollaries of Completeness

(2) First Order Compactness Theorem.

An infinite set of first order sentences Φ is unsatisfiable if and only if some finite subset of Φ is unsatisfiable

Proof Let A be the empty sequent (or any unsatisfiable formula)

Φ unsatisfiable means $\Phi \models A$.

Thus (by completeness) there is a Φ-LK proof of A

proof. Thus there is a finite subset Φ' of Φ such that there is a Φ'-LK proof of A

$\therefore$ Φ' is unsatisfiable.

(other direction is easy)

Dealing with Equality

So far we have treated equality predicate as true equality. We want to show that a finite number of equality axioms essentially characterizes true equality

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Definition A **weak $\mathcal{L}$-structure** is an $\mathcal{L}$-structure where $=$ can be any binary predicate

Question: Can we define a finite set of sentences $\mathcal{E}$ that defines equality? (That is, a proper structure satisfies $\mathcal{E}$ and any weak structure satisfying $\mathcal{E}$ must have $=$ be true equality?)

Dealing with Equality

Question: Can we define a finite set of sentences $\mathcal{E}$ that defines equality? (That is, a proper structure satisfies $\mathcal{E}$ and any weak structure satisfying $\mathcal{E}$ must have $=$ be true equality?)

No! Let $M' = M \cup \{m'\}$ New element

Fix some $m \in M$, and let $m \overset{\mathfrak{M}'}{=} m'$

and otherwise $\mathfrak{M}'$ on m' behaves like $\mathfrak{M}$ on m

Dealing with Equality

<u>Question</u>: Can we define a finite set of sentences $\mathcal{E}$ that defines equality? (That is, a proper structure satisfies $\mathcal{E}$ and any weak structure satisfying $\mathcal{E}$ must have $=$ be true equality?)

But this is the only counterexample.
There is a natural, finite set of axioms that characterizes true equality (up to isomorphism)

Dealing with Equality

Equality Axioms for $\mathcal{L}$ ($\mathcal{E}_{\mathcal{L}}$)

= is an equiv reln

E1. $\forall x (x = x)$

E2. $\forall x \forall y (x = y \supset y = x)$

E3. $\forall x \forall y \forall z ((x = y \wedge y = z) \supset x = z)$

E4. $\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n (x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \supset f x_1 \ldots x_n = f y_1 \ldots y_n$

for all n-ary function symbols, and for all $n \geq 1$

E5. $\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n ((x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \supset (P x_1 \ldots x_n \supset P y_1 \ldots y_n))$

equivalence relation preserved by functions and predicates

Equality Theorem

Theorem Let Φ be a set of $\mathcal{L}$-sentences. Φ is satisfiable iff $\Phi \cup \mathcal{E}_{\mathcal{L}}$ is satisfied by some weak $\mathcal{L}$-structure.

Proof straightforward (see Lecture Notes)

LK with Equality

Add these axioms for all terms $u, t, u_1, \ldots, t_1 \ldots$

L1 $\longrightarrow t = t$

L2 $t = u \longrightarrow u = t$

L3 $t = u, u = v \longrightarrow t = v$

L4 $t_1 = u_1, \ldots, t_n = u_n \longrightarrow f t_1 \ldots t_n = f u_1 \ldots u_n$

L5 $t_1 = u_1, \ldots, t_n = u_n, P t_1 \ldots t_n \longrightarrow P u_1 \ldots u_n$

Now an LK-Φ proof of $\longrightarrow A$ means an LK proof of A from Φ and from above axioms