

Announcements

- HW1 DUE OCT 11
- Test 1 Wed OCT 19 (in class)

This Week

- HW 1 - clarifications + some examples
- Proof Systems for First order Logic
LK : Soundness and Completeness

Q1.

DP procedure Input : $f = C_1 \wedge \dots \wedge C_m$ over x_1, \dots, x_n
Output : A Resolution Refutation of f (if UNSAT)

Initially $\mathcal{C}_1 := \{ C \mid C \text{ is a clause of } f \}$

For $i=1, \dots, n$:

- Loop:
If \exists 2 clauses in \mathcal{C}_i of form $(x_i \vee D), (\bar{x}_i \vee E)$
such that $(D \vee E) \notin \mathcal{C}_i$, add $(D \vee E)$ to \mathcal{C}_i
Else end loop
- $\mathcal{C}_{i+1} := \{ C \mid C \in \mathcal{C}_i \text{ and } C \text{ does not contain } x_i \}$

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- $\mathcal{C}_{i+1} := \{ C \mid C \in \mathcal{C}_i \text{ and } C \text{ does not contain } x_i \}$

* Observe \mathcal{C}_i only contains the variables x_i, x_{i+1}, \dots, x_n

Q1.

DP procedure Input : $f = C_1 \wedge \dots \wedge C_m$ over x_1, \dots, x_n
Output : A Resolution Refutation of f (if UNSAT)

Example: $f = (x_1 \vee x_2 \vee \bar{x}_3) (\bar{x}_1 \vee x_2 \vee x_4) (x_1 \vee x_2 \vee x_3) (\bar{x}_2 \vee \bar{x}_3) (\bar{x}_2 \vee x_4) (\bar{x}_4)$

① Resolve on x_1 : $(x_2 \vee x_4 \vee \bar{x}_3), (x_2 \vee x_3 \vee x_4)$

$C_2 = \{ (x_2 \vee x_4 \vee \bar{x}_3), (x_2 \vee x_3 \vee x_4), (\bar{x}_2 \vee \bar{x}_3), (\bar{x}_2 \vee x_4), (\bar{x}_4) \}$

② Resolve on x_2 : $(\bar{x}_3 \vee x_4), (x_3 \vee x_4)$

$C_3 = \{ (\bar{x}_3 \vee x_4), (x_3 \vee x_4), (\bar{x}_4) \}$

③ Resolve on x_3 : (x_4)

$C_4 = \{ (\bar{x}_4), (\bar{x}_4) \}$

④ Resolve on x_4 : \emptyset $C_5 = \{ \emptyset \}$

Q1.

- (a) apply DP procedure to given CNF formula
- (b) show DP procedure runs in time polynomial in m whenever f is a 2CNF formula.
- (c) Prove completeness of DP procedure

Idea Prove by induction on # steps ($i=1..n$)
that if \mathcal{C}_i is unsatisfiable, then
 \mathcal{C}_{i+1} is unsatisfiable

Q2.

Prove that ^{tree-like} Resolution refutations are closed under restrictions

Let $f = C_1 \wedge \dots \wedge C_m$ be UNSAT CNF, over x_1, \dots, x_n

Let Π be a ^{tree-like} resolution refutation of f

Let ρ set $x_i \leftarrow b$, $b \in \{0, 1\}$.

Let $\Pi|_{\rho}$ be the sequence of clauses obtained from Π by setting $x_i \leftarrow b$

[if $\rho(x_i) = 0$, clauses $(x_i \vee D) \rightarrow (D)$
clauses $(\bar{x}_i \vee E) \rightarrow (E)$
clauses without x_i stay same
similarly for $\rho(x_i) = 1$

Q2.

Prove that ^{tree-like} Resolution refutations are closed under restrictions

Let $\Pi|_p$ be the sequence of clauses obtained from Π by setting $x_i \leftarrow b$

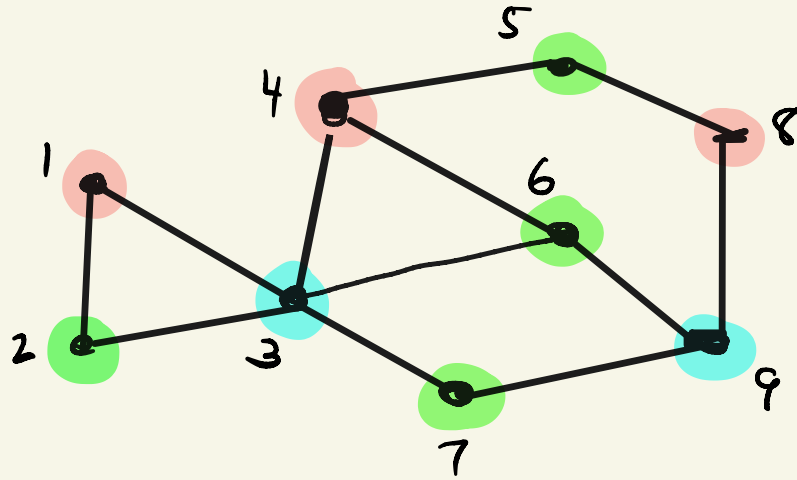
[if $p(x_i) = 0$, clauses $(x_i \vee D) \rightarrow (D)$
clauses $(\bar{x}_i \vee E) \rightarrow (E)$
clauses without x_i stay same
similarly for $p(x_i) = 1$

Show: $\Pi|_p$ can be turned into a tree-like Resolution refutation of $f|_p$, of size $\leq |\Pi|$

Q4.

Let $g = (V, E)$ be an undirected graph

Example:



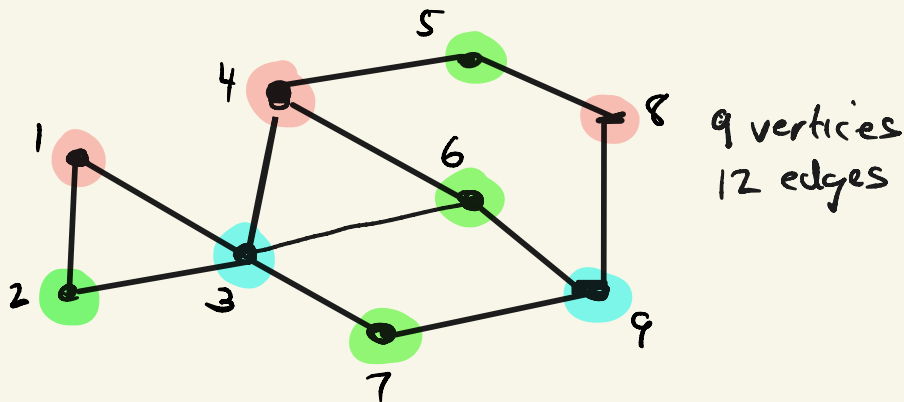
$$|V| = 9$$
$$|E| = 12$$

$$\chi : \begin{array}{l} V_1, V_4, V_8 \rightarrow \text{pink} \\ V_2, V_6, V_7 \rightarrow \text{green} \\ V_3, V_9 \rightarrow \text{cyan} \end{array}$$

Q4.

Let $G = (V, E)$ be an undirected graph

Example:



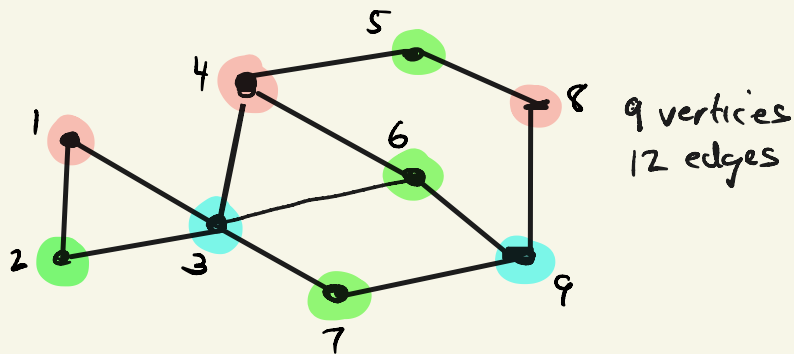
For the above graph G ($n=9, m=12$),

Give a propositional formula $3COLOR_G$, using variables $\{R_i, B_i, G_i \mid i=1, \dots, 9\}$ such that

$3COLOR_G$ is satisfiable if and only if G is 3-colorable

Q4.

Example:



More generally, give an efficient algorithm A that takes as input (n, g) , such that $n > 0$, and g is an n -vertex graph, and outputs

a propositional formula $3COLOR_g$, using variables $\{R_i, B_i, G_i \mid i=1, \dots, n\}$ such that

$3COLOR_g$ is satisfiable if and only if g is 3-colorable

Q6.

Let $\Phi = \{A_1, A_2, A_3, \dots\}$ be a countably infinite set of \mathcal{L} -sentences. Let $\Phi \models B$

Prove that $\exists n$ such that $B \models A_n$

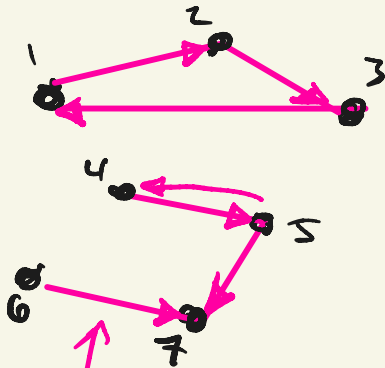
can assume

$\forall i \quad A_1, \dots, A_i \not\models A_{i+1}$

Q7. Give a sentence A of first order Logic over $\mathcal{L} = \{ ; R, = \}$ where R is a 2-ary predicate symbol such that for any finite model $\mathcal{M} = (M, \hat{R})$, $\mathcal{M} \models A$ (\mathcal{M} satisfies A) iff \hat{R} corresponds to a disjoint union of directed cycles.

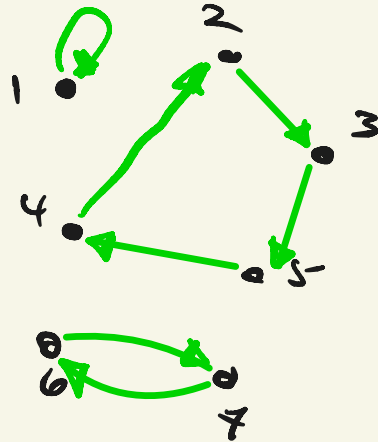
Example 1

$M = \{1, 2, 3, \dots, 7\}$



$\hat{R}(6,7) = 1$

Example 2



Q8 (Extra credit)

$$\neg PH P_3^4$$

Variables: $P_{i,j}$

Clauses: $(P_{1,1} \vee P_{1,2} \vee P_{1,3})$
 $(P_{2,1} \vee P_{2,2} \vee P_{2,3})$
 $(P_{3,1} \vee P_{3,2} \vee P_{3,3})$
 $(P_{4,1} \vee P_{4,2} \vee P_{4,3})$

} Pigeon clauses

$\forall j \in \{1, 2, 3\} : (\neg P_{1,j} \vee \neg P_{2,j}), (\neg P_{1,j} \vee \neg P_{3,j}), (\neg P_{1,j} \vee \neg P_{4,j})$
 $(\neg P_{2,j} \vee \neg P_{3,j}), (\neg P_{2,j} \vee \neg P_{4,j}), (\neg P_{3,j} \vee \neg P_{4,j})$

} Hole clauses

expresses: there is a 1-1, onto map from $\{1, 2, 3, 4\} \rightarrow \{1, 2, 3\}$

Q8.

$\neg \text{PHP}_n^{n+1}$: ① $\forall i \in \{1, 2, \dots, n+1\} : (P_{i,1} \vee P_{i,2} \vee \dots \vee P_{i,n})$

② $\forall i_1, i_2 \in \{1, \dots, n+1\} \ i_1 \neq i_2, \forall j \in \{1, \dots, n\}$
 $(\neg P_{i_1,j} \vee \neg P_{i_2,j})$

Prove: any tree-like Res Ref. of $\neg \text{PHP}_n^{n+1}$
requires size $\geq 2^n$

FIRST ORDER SEQUENT CALCULUS LK

Lines are again **sequents**

$$A_1, \dots, A_k \rightarrow B_1, \dots, B_l$$

where each A_i, B_j is a proper \mathcal{L} -formula

RULES

OLD RULES OF PK

PLUS NEW RULES FOR \forall, \exists

like a large
AND

like a large
OR

New Logical Rules for \forall, \exists

$$\forall\text{-left} \quad \frac{A(t), \Gamma \rightarrow \Delta}{\forall x A(x), \Gamma \rightarrow \Delta}$$

$$\forall\text{-Right} \quad \frac{\Gamma \rightarrow \Delta, A(b)}{\Gamma \rightarrow \Delta, \forall x A(x)}$$

$$\exists\text{-left} \quad \frac{A(b), \Gamma \rightarrow \Delta}{\exists x A(x), \Gamma \rightarrow \Delta}$$

$$\exists\text{-right} \quad \frac{\Gamma \rightarrow \Delta, A(t)}{\Gamma \rightarrow \Delta, \exists x A(x)}$$

* A, t are proper

* b is a free variable not appearing in lower sequent of rule

Example of an LK proof

$$Pa \rightarrow Pa$$

$$Pa, Qa \rightarrow Pa$$

AND-left

$$Pa \wedge Qa \rightarrow Pa$$

\exists -Rt

$$Pa \wedge Qa \rightarrow \exists x Px$$

\exists Left

$$\exists x (Px \wedge Qx) \rightarrow \exists x Px$$

AND-Rt

$$\exists x (Px \wedge Qx) \rightarrow \exists x Px \wedge \exists x Qx$$

$$Qa \rightarrow Qa$$

$$Pa, Qa \rightarrow Qa$$

AND-left

$$Pa \wedge Qa \rightarrow Qa$$

\exists -Rt

$$Pa \wedge Qa \rightarrow \exists x Qx$$

\exists -Left

$$\exists x (Px \wedge Qx) \rightarrow \exists x Qx$$

$$\underbrace{\forall x A(x)} \vee \underbrace{\neg \forall x A(x)}_{\exists y \neg A(y)}$$

Is this formula satisfiable?

$M = \{1, 2\}$
 $\hat{A}(1) = \text{true}$
 $\hat{A}(2) = \text{true}$

	A
1	T
2	T
3	F
⋮	F
⋮	T
⋮	⋮
⋮	⋮
⋮	⋮
⋮	⋮

$$\forall x_1 \forall x_2 \exists x_3 \left(\text{Plus}(x_1, x_2) = x_3 \wedge \text{GEQ}(x_3, x_1) \right)$$

$$\mathcal{M} = \left(\underbrace{\mathbb{N}}_M, \text{"true +"} \right) \text{ satisfies}$$

$$\mathcal{M} = \left(\mathbb{N}, \widehat{\text{Plus}}(x, y) = 0 \quad \forall x, y \right)$$

$$\text{GEQ} = \text{"true"} \cong$$

SOUNDNESS

Defn A first order sequent $A_1, \dots, A_k \rightarrow B_1, \dots, B_\ell$ is **valid** if and only if its associated formula $(A_1 \wedge \dots \wedge A_k) \supset (B_1 \vee \dots \vee B_\ell)$ is valid.

Soundness Theorem for LK Every sequent provable in LK is valid

Soundness (Proof) : By induction on the number of sequents in proof

Key Lemma For all rules/axioms of LK

For all instantiations of a rule/axiom

if upper sequents of the rule are valid
then lower sequent is also valid

$$\frac{\Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, A \vee B} \quad A \rightarrow A$$

Key Lemma (Proof sketch)

Example: \forall Right Rule $\left(\frac{\Gamma \rightarrow A(b), \Delta}{\Gamma \rightarrow \forall x A(x)}, \Delta \right)$

Assume: $\Gamma \rightarrow A(b), \Delta$ is valid. Show $\Gamma \rightarrow \forall x A(x), \Delta$ is also valid

Let $\Gamma = B_1, \dots, B_k$, $\Delta = C_1, \dots, C_e$

Then $\underbrace{B_1 \vee B_2 \vee \dots \vee B_k \vee C_1 \vee \dots \vee C_e}_{D} \vee A(b)$ is valid

ie $\forall \mathcal{M}, \mathcal{G} : \mathcal{M} \models D \vee A(b)$ [6]

Show $\forall \mathcal{M}, \mathcal{G} : \mathcal{M} \models D \vee \forall x A(x)$ [6]

Key Lemma (Proof sketch)

Example: \forall Right Rule $\left(\frac{\Gamma \rightarrow A(b), \Delta}{\Gamma \rightarrow \forall x A(x), \Delta} \right)$

Assume: $\Gamma \rightarrow A(b), \Delta$ is valid. Show $\Gamma \rightarrow \forall x A(x), \Delta$ is also valid

Let $\Gamma = B_1, \dots, B_k$, $\Delta = C_1, \dots, C_e$

Then $\underbrace{B_1 \vee \dots \vee B_k \vee C_1 \vee \dots \vee C_e}_{D} \vee A(b)$ is valid

ie $\forall \mathcal{M}, \mathcal{G} : \mathcal{M} \models D \vee A(b) [\mathcal{G}]$

Let \mathcal{G}' be an object assignment to all free variables in $D \vee A(b)$ except for b

Case 1: $\mathcal{M} \models D [\mathcal{G}']$

$\therefore \mathcal{M} \models D \vee A(b) [\mathcal{G}] \forall \mathcal{G}$ extending \mathcal{G}' (since b does not occur in D)

Case 2 $\mathcal{M} \models D [\mathcal{G}']$.

$\therefore \mathcal{M} \models A(b) [\mathcal{G}', m/b] \forall m \in M$ (since $D \vee A(b)$ is valid)

$\therefore \mathcal{M} \models A(b) \forall \mathcal{G}$ extending \mathcal{G}'

$\therefore \mathcal{M} \models D \vee A(b) [\mathcal{G}]$

Next: Gödel's COMPLETENESS THEOREM

Defn An $LK-\Phi$ proof is an LK -proof, but leaves are either axioms $(A \rightarrow A)$ or of the form $\rightarrow A$ for $A \in \Phi$

goal prove that if $\Gamma \rightarrow \Delta$ is a logical consequence of Φ , then there is an $LK-\Phi$ proof of $\Gamma \rightarrow \Delta$ (called **Derivational completeness**)

Defn Let $A(a_1 \dots a_n)$ be a formula with free variables $a_1 \dots a_n$. Then $\forall A$ is $\forall x_1 \forall x_2 \dots \forall x_n A(x_1 \dots x_n)$ (called **universal closure of A**)