Announcements

- HW1 DUE OCT 11

- Test 1 Wed OCT 19 (in class)

This Week

- HW 1 – clarifications + some examples

- Proof Systems for First order Logic

  LK : Soundness and Completeness

Q1.

DP procedure

Input : $f = C_1 \wedge \dots \wedge C_m$ over $x_1, \dots, x_n$

Output : A Resolution Refutation of f (if UNSAT)

Initially $\mathcal{C}_1 := \{ C \mid C \text{ is a clause of } f \}$

For $i = 1, \dots, n$:

- Loop:
  If $\exists$ 2 clauses in $\mathcal{C}_i$ of form $(x_i \vee D), (\bar{x}_i \vee E)$
  such that $(D \vee E) \notin \mathcal{C}_i$, add $(D \vee E)$ to $\mathcal{C}_i$
  Else end Loop
- $\mathcal{C}_{i+1} := \{ C \mid C \in \mathcal{C}_i \text{ and } C \text{ does not contain } x_i \}$

Q1.

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- $\mathcal{C}_{i+1} := \{ C \mid C \in \mathcal{C}_i \text{ and } C \text{ does not contain } x_i \}$

* Observe C_i only contains the variables $x_i, x_{i+1}, \dots, x_n$

Q1.

DP procedure — Input : $f = C_1 \wedge \cdots \wedge C_m$ over $x_1, \ldots, x_n$

Output : A Resolution Refutation of f (if UNSAT)

Example: $f = (x_1 \vee x_2 \vee \bar{x}_3)(\bar{x}_1 \vee x_2 \vee x_4)(x_1 \vee x_2 \vee x_3)(\bar{x}_2 \vee \bar{x}_3)(\bar{x}_2 \vee x_4)(\bar{x}_4)$

① Resolve on x_1 : $(x_2 \vee x_4 \vee \bar{x}_3), (x_2 \vee x_3 \vee x_4)$

$C_2 = \{ (x_2 \vee x_4 \vee \bar{x}_3), (x_2 \vee x_3 \vee x_4), (\bar{x}_2 \vee \bar{x}_3), (\bar{x}_2 \vee x_4), (\bar{x}_4) \}$

② Resolve on x_2 : $(\bar{x}_3 \vee x_4), (x_3 \vee x_4)$

$C_3 = \{ (\bar{x}_3 \vee x_4), (x_3 \vee x_4), (\bar{x}_4) \}$

③ Resolve on x_3 : (x_4)

$C_4 = \{ (x_4), (\bar{x}_4) \}$

④ Resolve on x_4 : ϕ $\qquad C_5 = \{ \phi \}$

Q1.

(a) apply DP procedure to given CNF formula

(b) show DP procedure runs in time polynomial in m whenever f is a 2CNF formula.

(c) Prove completeness of DP procedure

Idea: Prove by induction on # steps $(i = 1 \ldots n)$ that if e_i is unsatisfiable, then e_{i+1} is unsatisfiable

Q2.

Prove that tree-like Resolution refutations are closed under restrictions

Let $f = C_1 \wedge \dots \wedge C_m$ be UNSAT CNF, over $x_1 \dots x_n$

Let Π be a tree-like resolution refutation of f

Let ρ set $x_i \leftarrow b$, $b \in \{0,1\}$.

Let $\Pi|_\rho$ be the sequence of clauses obtained from Π by setting $x_i \leftarrow b$

- if $\rho(x_i) = 0$, clauses $(x_i \vee D) \longrightarrow (D)$
  - clauses $(\bar{x}_i \vee E) \longrightarrow (1)$
  - clauses without x_i stay same
- similarly for $\rho(x_i) = 1$

Q2.

Prove that tree-like Resolution refutations are closed under restrictions

Let $\Pi|_\rho$ be the sequence of clauses obtained from Π by setting $x_i \leftarrow b$

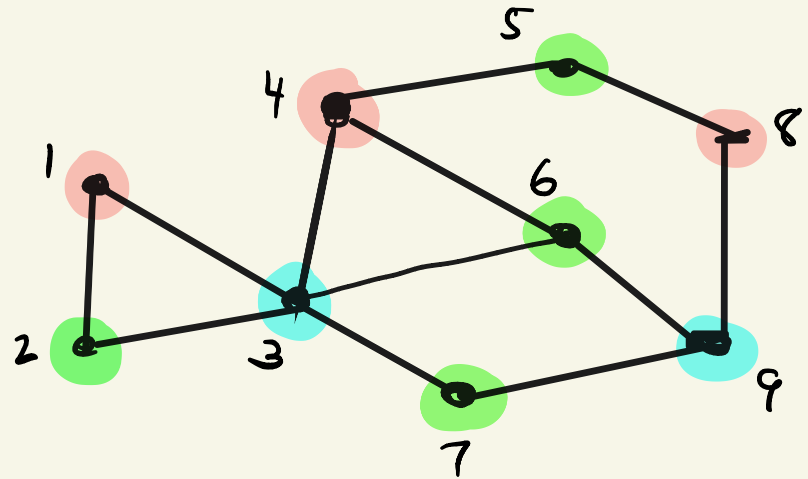
$\left[\begin{array}{ll} \text{if } \rho(x_i)=0, & \text{clauses } (x_i \vee D) \longrightarrow (D) \\ & \text{clauses } (\bar{x}_i \vee E) \longrightarrow (1) \\ & \text{clauses without } x_i \text{ stay same} \\ \text{similarly for } \rho(x_i)=1 & \end{array}\right.$

Show: $\Pi|_\rho$ can be turned into a tree-like Resolution Refutation of $f|_\rho$, of size $\le |\Pi|$

Q4.

Let $G = (V, E)$ be an undirected graph

Example:



$|V| = 9$

$|E| = 12$

$\chi : v_1, v_4, v_8 \rightarrow$

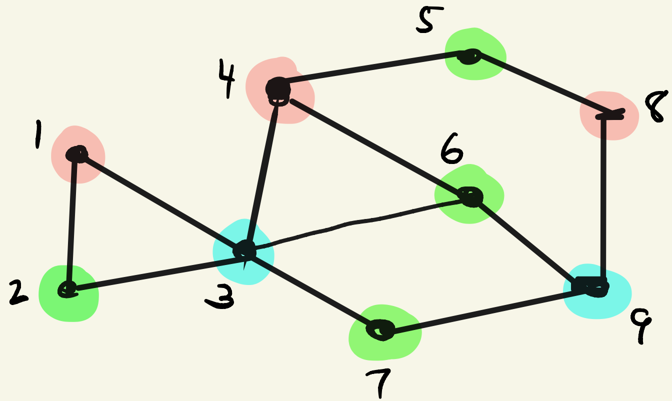
$v_2, v_6, v_7 \rightarrow$

$v_3, v_9 \rightarrow$

Q4.

Let $G=(V,E)$ be an undirected graph

Example:



9 vertices
12 edges

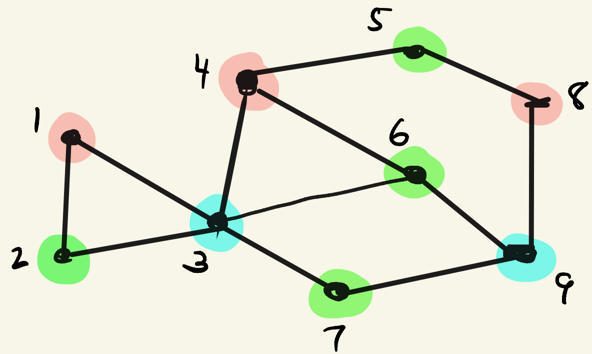
For the above graph G $(n=9,\ m=12)$,

give a propositional formula $3COLOR_G$, using variables $\{R_i, B_i, G_i \mid i=1,\dots,9\}$ such that

$3COLOR_G$ is satisfiable if and only if G is 3-colorable

Q4.

Example:



9 vertices
12 edges

More generally, give an efficient algorithm A that takes as input (n, g), such that $n > 0$, and g is an n-vertex graph, and outputs a propositional formula $3COLOR_g$, using variables $\{R_i, B_i, G_i \mid i = 1, \ldots, n\}$ such that

$3COLOR_g$ is satisfiable if and only if g is 3-colorable

Q6.

Let $\Phi = \{A_1, A_2, A_3, \ldots\}$ be a countably infinite set of $\mathcal{L}$-sentences. Let $\Phi \models B$

Prove that $\exists n$ such that $B \not\models A_n$

and assume

$\forall i \quad A_1 \ldots A_i \not\models A_{i+1}$

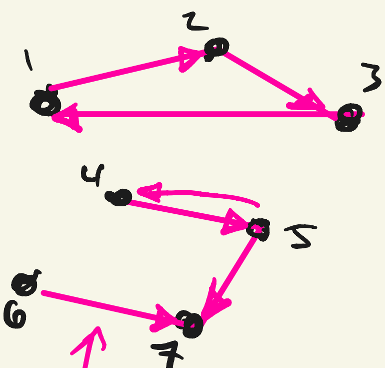
Q7. Give a sentence A of first order logic over $\mathcal{L} = \{ ; R, = \}$ where R is a 2-ary predicate symbol such that for any finite model $\mathcal{M} = (M, \hat{R})$,

$\mathcal{M} \models A$ ($\mathcal{M}$ satisfies A) iff

$\hat{R}$ corresponds to a disjoint union of directed cycles.

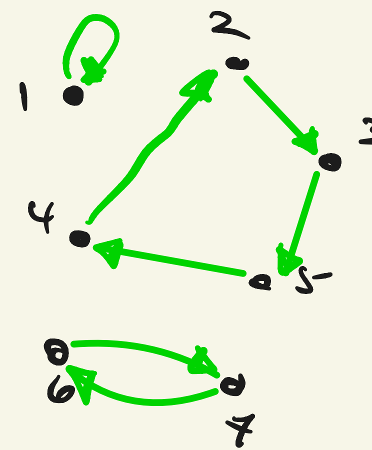
Example 1

$M = \{1, 2, 3, .. 7\}$



$\hat{R}(6,7) = 1$

Example 2



Q8 (Extra Credit)

$$\boxed{\neg PHP^4_3}$$

Variables: $P_{i,j}$

Clauses:
$$\left.\begin{array}{l}(P_{11} \vee P_{12} \vee P_{13})\\(P_{2,1} \vee P_{2,2} \vee P_{2,3})\\(P_{3,1} \vee P_{3,2} \vee P_{3,3})\\(P_{4,1} \vee P_{4,2} \vee P_{4,3})\end{array}\right\}\text{ Pigeon clauses}$$

$$\left.\begin{array}{l}\forall j \in \{1,2,3\} : (\neg P_{1j} \vee \neg P_{2j}), (\neg P_{1,j} \vee \neg P_{3j}) (\neg P_{1j} \vee \neg P_{4,j})\\(\neg P_{2j} \vee \neg P_{3j}), (\neg P_{2j} \vee \neg P_{4j}) (\neg P_{3j} \vee \neg P_{4j})\end{array}\right\}\text{ Hole clauses}$$

expresses: there is a 1-1, onto map from $\{1,2,3,4\} \to \{1,2,3\}$

Q8.

$\neg PHP_n^{n+1}$: ① $\forall i \in \{1, 2, .. n+1\}$: $(P_{i,1} \vee P_{i,2} \vee .. \vee P_{i,n})$

② $\forall i_1, i_2 \in \{1, .., n+1\}$ $i_1 \neq i_2$, $\forall j \in \{1, .., n\}$

$(\neg P_{i_1 j} \vee \neg P_{i_2 j})$

Prove: any tree-like Res Ref. of $\neg PHP_n^{n+1}$ requires size $\geq 2^n$

FIRST ORDER SEQUENT CALCULUS LK

Lines are again sequents

$$A_1, \ldots, A_k \rightarrow B_1, \ldots, B_\ell$$

where each A_i, B_j is a proper $\mathcal{L}$-formula

RULES

OLD RULES OF PK

PLUS NEW RULES FOR $\forall$, $\exists$

like a large AND

Large OR

New Logical Rules for $\forall$, $\exists$

$\forall$-left
$$\frac{A(t), \Gamma \rightarrow \Delta}{\forall x\, A(x), \Gamma \rightarrow \Delta}$$

$\forall$-Right
$$\frac{\Gamma \rightarrow \Delta, A(b)}{\Gamma \rightarrow \Delta, \forall x\, A(x)}$$

$\exists$-left
$$\frac{A(b), \Gamma \rightarrow \Delta}{\exists x\, A(x), \Gamma \rightarrow \Delta}$$

$\exists$-right
$$\frac{\Gamma \rightarrow \Delta, A(t)}{\Gamma \rightarrow \Delta, \exists x\, A(x)}$$

* A, t are proper
* b is a free variable Not appearing in lower sequent of rule

Example of an LK proof

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{Pa \to Pa}{Pa, Qa \to Pa}}{Pa \wedge Qa \to Pa}\,\text{AND-left}}{Pa \wedge Qa \to \exists x Px}\,\exists\text{-Rt}}{\exists x (Px \wedge Qx) \to \exists x Px}\,\exists\text{ Left}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{Qa \to Qa}{Pa, Qa \to Qa}}{Pa \wedge Qa \to Qa}\,\text{AND-Left}}{Pa \wedge Qa \to \exists x Qx}\,\exists\text{-Rt}}{\exists x (Px \wedge Qx) \to \exists x Qx}\,\exists\text{-Left}
}{
\exists x (Px \wedge Qx) \to \exists x Px \wedge \exists x Qx
}\,\text{AND-RT}
$$

$$\overbrace{\forall x\, A(x)} \lor \overbrace{\exists y\, \neg A(y)}^{\neg \forall x\, A(x)}$$

Is this formula satisfiable?

$$\mathfrak{M} = \begin{cases} M = \{1, 2\} \\ \hat{A}(1) = \cancel{\text{false}}\ \text{true} \\ \hat{A}(2) = \text{true} \end{cases}$$

M

	A		
1	T		
2	T		
3	F		
.	F		
.	T		
⋮	⋮		

$$\overline{\forall x_1 \forall x_2 \exists x_3 \left(\text{Plus}(x_1, x_2) = x_3 \quad \wedge \quad \text{GEQ}(x_3, x_1) \right)}$$

$$\mathfrak{M} = \Big(\underbrace{\mathbb{N}}_{M} , \text{"true +"} \Big) \quad \text{satisfies}$$

$$\mathfrak{M} = \Big(\mathbb{N}, \ \widehat{\text{Plus}}(x, y) = 0 \quad \forall x, y \Big)$$

$$\text{GEQ} = \text{"true"} \geq$$

SOUNDNESS

Defn A first order sequent $A_1, \ldots, A_k \to B_1, \ldots, B_\ell$ is **valid** if and only if its associated formula $(A_1 \wedge \ldots \wedge A_k) \supset (B_1 \vee \ldots \vee B_\ell)$ is valid.

Soundness Theorem for LK Every sequent provable in LK is valid

Soundness (Proof) : By induction on the number of sequents in proof

Key Lemma For all rules/axioms of LK

For all instantiations of a rule/axiom

if upper sequents of the rule are valid

then lower sequent is also valid

$$\frac{\Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, A \vee B}$$

$$A \rightarrow A$$

Key Lemma (Proof sketch)

Example: $\forall$Right Rule $\left(\dfrac{\Gamma \to A(b), \Delta}{\Gamma \to \forall x\, A(x), \Delta} \right)$

Assume: $\Gamma \to A(b), \Delta$ is valid. Show $\Gamma \to \forall x\, A(x), \Delta$ is also valid

Let $\Gamma = B_1, \ldots, B_k$, $\Delta = C_1, \ldots, C_\ell$

Then $\underbrace{\neg B_1 \vee \neg B_2 \vee \ldots \vee \neg B_k \vee C_1 \vee \ldots \vee C_\ell}_{D} \vee A(b)$ is valid

ie $\forall \mathfrak{M}, \sigma : \mathfrak{M} \models D \vee A(b)\, [\sigma]$

Show $\forall \mathfrak{M}, \sigma : \mathfrak{M} \models D \vee \forall x\, A(x)\, [\sigma]$

Key Lemma (Proof sketch)

Example: $\forall$Right Rule $\left(\dfrac{\Gamma \to A(b), \Delta}{\Gamma \to \forall x\, A(x), \Delta}\right)$

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Let $\Gamma = B_1, \ldots, B_k$, $\Delta = C_1, \ldots, C_\ell$

Then $\underbrace{\neg B_1 \vee \neg B_2 \vee \ldots \vee \neg B_k \vee C_1 \vee \ldots \vee C_\ell}_{D} \vee A(b)$ is valid

ie $\forall \mathfrak{M}, \sigma : \mathfrak{M} \models D \vee A(b)\, [\sigma]$

Let σ' be an object assignment to all free variables in $D \vee A(b)$ except for b

Case 1: $\mathfrak{M} \models D[\sigma']$

$\therefore \mathfrak{M} \models D \vee A(b)[\sigma]\ \forall \sigma$ extending σ' (since b does not occur in D)

Case 2 $\mathfrak{M} \not\models D[\sigma']$.

$\therefore \mathfrak{M} \models A(b)[\sigma', m/b]\ \forall m \in M$ (since $D \vee A(b)$ is valid)

$\therefore \mathfrak{M} \models A(b)\ \forall \sigma$ extending σ'

$\therefore \mathfrak{M} \models D \vee A(b)[\sigma]$

TODAY: gödels COMPLETENESS THEOREM

<u>Defn</u> An LK-Φ proof is an LK-proof, but leaves are either axioms $(A \Rightarrow A)$ or of the form $\Rightarrow A$ for $A \in \Phi$

goal prove that if $\Gamma \Rightarrow \Delta$ is a logical consequence of Φ, then there is an LK-Φ proof of $\Gamma \Rightarrow \Delta$
(called Derivational completeness)

<u>Defn</u> Let $A(a_1 .. a_n)$ be a formula with free variables $a_1 .. a_n$. Then $\forall A$ is $\forall x_1 \forall x_2 .. \forall x_n A(x_1 .. x_n)$
(called universal closure of A)

TODAY: LK COMPLETENESS

(MAIN LEMMA) Completeness Lemma

If $\Gamma \Rightarrow \Delta$ is a logical consequence of a set of (possibly infinite) formulas $\forall \Phi$ then there exists a finite subset $\{C_1, \ldots, C_n\}$ of Φ such that

$\forall C_1, \ldots, \forall C_n, \Gamma \Rightarrow \Delta$ has a (cut-free) PK proof

* We will assume = not in Language for now

Derivational Completeness Theorem

Let Φ be a set of sequents or formulas such that the sequent $\Gamma \to \Delta$ is a logical consequence of $\forall \Phi$.

Then there is an LK-Φ proof of $\Gamma \to \Delta$.

Proof follows from Completeness Lemma
(similar to derivational completeness of PK from completeness)

Proof of LK Completeness Lemma

High Level idea (assume Φ is empty for now)

- As in PK completeness, we want to construct an LK proof in reverse.
- Start with $\Gamma \Rightarrow \Delta$ at root, and apply rules in reverse (to break up a formula into one or 2 smaller ones)
- Tricky rules are $\exists$right & $\forall$left.
  When applying one of these in reverse, Need to "guess" a term

New Logical Rules for $\forall$, $\exists$

$\forall$-left $$\frac{A(t), \Gamma \to \Delta}{\forall x\, A(x), \Gamma \to \Delta}$$

$\forall$-Right $$\frac{\Gamma \to \Delta, A(b)}{\Gamma \to \Delta, \forall x\, A(x)}$$

$\exists$-left $$\frac{A(b), \Gamma \to \Delta}{\exists x\, A(x), \Gamma \to \Delta}$$

$\exists$-right $$\frac{\Gamma \to \Delta, A(t)}{\Gamma \to \Delta, \exists x\, A(x)}$$

* A, t are proper
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Proof of LK Completeness Lemma

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- As in PK completeness, we want to construct an LK proof in reverse.
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- Tricky rules are $\exists$right + $\forall$left. When applying one of these in reverse, Need to "guess" a term

- Key is to systematically try all possible terms — without going down a rabbit hole.

Example of an LK proof

$$
? \qquad
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{Pa \to Pa}{Pa, Qa \to Pa}}{Pa \land Qa \to Pa}}{Pa \land Qa \to \exists x Px}}{\exists x(Px \land Qx) \to \exists x Px}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{Qa \to Qa}{Pa, Qa \to Qa}}{Pa \land Qa \to Qa}}{Pa \land Qa \to \exists x Qx}}{\exists x(Px \land Qx) \to \exists x Qx}
}{\exists x(Px \land Qx) \to \exists x Px \land \exists x Qx}
\qquad ?
$$

Example of an LK proof

$$Pa, Qa \longrightarrow Pb$$

$$Pa \wedge Qa \longrightarrow Pb$$

$$Pa \wedge Qa \longrightarrow \exists x Px \qquad\qquad Pa \wedge Qa \longrightarrow \exists x Qx$$

$$\exists x (Px \wedge Qx) \longrightarrow \exists x Px \qquad\qquad \exists x (Px \wedge Qx) \longrightarrow \exists x Qx$$

$$\exists x (Px \wedge Qx) \longrightarrow \exists x Px \wedge \exists x Qx$$

Instead:

$$Pa, Qa \rightarrow Pb, \exists x Px$$

$$Pa \wedge Qa \rightarrow Pb, \exists x Px$$

$$\dfrac{\dfrac{Pa \wedge Qa \rightarrow \exists x Px}{\exists x (Px \wedge Qx) \rightarrow \exists x Px} \qquad \dfrac{Pa \wedge Qa \rightarrow \exists x Qx}{\exists x (Px \wedge Qx) \rightarrow \exists x Qx}}{\exists x (Px \wedge Qx) \rightarrow \exists x Px \wedge \exists x Qx}$$

Instead

Try again

$$Pa, Qa \rightarrow Pb, Pfa, \exists x Px$$

$$Pa, Qa \rightarrow Pb, \exists x Px$$

$$Pa \wedge Qa \rightarrow Pb, \exists x Px$$

$$\dfrac{\dfrac{Pa \wedge Qa \rightarrow \exists x Px}{\exists x (Px \wedge Qx) \rightarrow \exists x Px} \qquad \dfrac{Pa \wedge Qa \rightarrow \exists x Qx}{\exists x (Px \wedge Qx) \rightarrow \exists x Qx}}{\exists x (Px \wedge Qx) \rightarrow \exists x Px \wedge \exists x Qx}$$

Instead

and again

and again

$$Pa, Qa \rightarrow Pb, Pfa, Pfb, \exists x Px$$

Try again

$$\dfrac{Pa, Qa \rightarrow Pb, Pfa, \exists x Px}{Pa, Qa \rightarrow Pb, \exists x Px}$$

$$Pa \wedge Qa \rightarrow Pb, \exists x Px$$

$$\dfrac{\dfrac{Pa \wedge Qa \rightarrow \exists x Px}{\exists x (Px \wedge Qx) \rightarrow \exists x Px} \qquad \dfrac{Pa \wedge Qa \rightarrow \exists x Qx}{\exists x (Px \wedge Qx) \rightarrow \exists x Qx}}{\exists x (Px \wedge Qx) \rightarrow \exists x Px \wedge \exists x Qx}$$

Instead

There are infinitely many choices! Need a systematic way to try all

and again

and again

$$Pa, Qa \Rightarrow Pb, Pfa, Pfb, \exists x Px$$

$$Pa, Qa \Rightarrow Pb, Pfa, \exists x Px$$

Try again

$$Pa, Qa \Rightarrow Pb, \exists x Px$$

$$Pa \land Qa \Rightarrow Pb, \exists x Px$$

$$\dfrac{\dfrac{Pa \land Qa \Rightarrow \exists x Px}{\exists x (Px \land Qx) \Rightarrow \exists x Px} \qquad \dfrac{Pa \land Qa \Rightarrow \exists x Qx}{\exists x (Px \land Qx) \Rightarrow \exists x Qx}}{\exists x (Px \land Qx) \Rightarrow \exists x Px \land \exists x Qx}$$

Instead

and again

and again

$$Pa, Qa \to Pb, Pfa, Pfb, \exists x Px$$

$$Pa, Qa \to Pb, Pfa, \exists x Px$$

Try again

$$Pa, Qa \to Pb, \exists x Px$$

$$Pa \wedge Qa \to Pb, \exists x Px$$

$$\dfrac{\dfrac{Pa \wedge Qa \to \exists x Px}{\exists x (Px \wedge Qx) \to \exists x Px} \qquad \dfrac{Pa \wedge Qa \to \exists x Qx}{\exists x (Px \wedge Qx) \to \exists x Qx}}{\exists x (Px \wedge Qx) \to \exists x Px \wedge \exists x Qx}$$

There are infinitely many choices! Need a systematic way to try all and for all frontier sequents in current proof!