Announcements

- HW1 out tomorrow

  (Due oct 11)

LAST CLASS

- Another proof system for propositional logic: PK
  - soundness of PK
  - completeness of PK
- Derivational soundness/completeness of PK
- Propositional compactness Theorem ← FINISH TODAY

Pages 9 – 17 of Lecture Notes

TODAY

- Propositional Compactness Theorem (FINISH)

- First Order Logic

  Language / Syntax

  Semantics : Models

(pages 18 - 27 of course notes)

Derivational Soundness & Completeness of PK

Definition Let Φ be a set of sequents, S a sequent

A PK-Φ proof of S is a PK-proof of S from Φ and axioms of PK.

(also written as $\Phi \vdash S$)

Theorem. Let Φ be a set of (possibly infinite) sequents. Then $\Phi \models S$ iff S has a (finite) PK-Φ proof

$$\Phi \models S \text{ iff } \Phi \vdash S$$

Propositional Compactness

Theorem (Form 2, see notes for 2 other equivalent forms)

Let Φ be a set of (possibly infinite) formulas

$\Phi \models A$ iff A is a logical consequence of a finite subset of Φ

We'll assume this for now and prove it after
Proof of 3 equivalent forms of Compactness as homework

Proof (Derivational Soundness/completeness)

By compactness, it suffices to prove the case where Φ is finite

- Let $\Phi = \{s_1, .., s_k\}$, and suppose $\Gamma \Rightarrow \Delta$ is a logical consequence of $\{s_1, .., s_k\}$. Thus

$$(*) \quad \Gamma, A_{s_1}, .., A_{s_k} \Rightarrow \Delta \quad \text{is valid}$$

- Thus by PK completeness, $(*)$ has a PK proof
- Derive $\Gamma \Rightarrow \Delta$ from $(*)$ and $\Rightarrow A_{s_1}, .., A_{s_k}$

Derive $\Gamma \to \Delta$ from $\{ \to A_{s_1}, \to A_{s_2}, \to A_{s_3}, (*) \}$

PK proof of

$$\Gamma, A_{s_1}, A_{s_2}, A_{s_3} \to \Delta$$

$$\frac{\to A_{s_1}}{\Gamma, A_{s_2}, A_{s_3} \to \Delta, A_{s_1}} \quad \text{(weakening)}$$

$$\frac{\Gamma, A_{s_1}, A_{s_2}, A_{s_3} \to \Delta \qquad \Gamma, A_{s_2}, A_{s_3} \to \Delta, A_{s_1}}{\Gamma, A_{s_2}, A_{s_3} \to \Delta} \quad \text{(cut)}$$

$$\frac{\to A_{s_2}}{\Gamma, A_{s_3} \to \Delta, A_{s_2}} \quad \text{(weakening)}$$

$$\frac{\hat{\Gamma}, A \to \hat{\Delta} \qquad \hat{\Gamma} \to A, \hat{\Delta}}{\Gamma \to \Delta}$$

$$\frac{\Gamma, A_{s_2}, A_{s_3} \to \Delta \qquad \Gamma, A_{s_3} \to \Delta, A_{s_2}}{\Gamma, A_{s_3} \to \Delta} \quad \text{(cut)}$$

$$\frac{\to A_{s_3}}{\Gamma \to A_{s_3}, \Delta} \quad \text{(weakening)}$$

$\Gamma \to \Delta$

$$\frac{\Gamma, A_{s_3} \to \Delta \qquad \Gamma \to A_{s_3}, \Delta}{\Gamma \to \Delta} \quad \text{(cut)}$$

Proof (Propositional Compactness)

Suppose $\Phi \vDash A$. Then $\overbrace{\Phi, \neg A}^{\Psi}$ is unsatisfiable

<u>show</u>: If Ψ is UNSAT, then some finite subset of Ψ is UNSAT (Form 1)

<u>Pf sketch</u> Assume the set of underlying atoms in Ψ is countable: $p_1, p_2, \ldots$

- Make decision tree T that queries p_1 at layer 1, then p_2 at layer 2, etc.

- Each path in T corresponds to a complete truth assignment

- Prune T to T':

  For every node v of T, remove subtree rooted below v if partial truth assignment from root to v falsifies some formula $f \in \varphi$. Label v by f

- Every path in T' is finite (since φ unsat, so $\forall$truth ass to all vars, some $f \in \varphi$ is falsified, and each $f \in \varphi$ is finite)

- By König's Lemma, T' is finite

<u>König's Lemma</u> If T' is a rooted binary tree, where every branch/path of T' is finite, then T' is finite.

- Thus, the formulas $\Psi' \subseteq \Psi$ labelling the leaves of T' form a finite subset of Ψ, and thus Ψ' is UNSAT & finite subset of Ψ.

FIRST ORDER LOGIC

Underlying language $\mathcal{L}$ specified by:

1. $\forall n \in \mathbb{N}$ a set of n-ary function symbols (ie.,: $f, g, h, +, \cdot$)
   0-ary function symbols are called constants

2. $\forall n \in \mathbb{N}$ a set of n-ary predicate symbols (i.e. $P, Q, R, <, \leq$)

Plus:

- Variables: $x, y, z, \ldots a, b, c, \ldots$
- $\neg, \vee, \wedge, \exists, \forall$
- parenthesis (,)

Built in symbols

Example 1 $\mathcal{L}_A$ (language of arithmetic)

$$\mathcal{L}_A = \{ \underbrace{\overset{\text{0-ary}}{0}, \overset{\text{1-ary}}{s}, \overset{\text{2-ary}}{+}, \overset{\text{2-ary}}{\cdot}}_{\text{function symbols}} ; \underbrace{\overset{\text{2-ary}}{=}}_{\text{relation symbols}} \}$$

0 constant (0-ary function symbol)

s unary function symbol

$+, \cdot$ binary function symbols

$=$ binary predicate symbol

Terms over $\mathcal{L}$

(1) Every variable is a term

(2) If f is an n-ary function symbol, and $t_1, \ldots, t_n$ terms, then $f t_1 \ldots t_n$ is a term

Terms over $\mathcal{L}$

(1) Every variable is a term

(2) If f is an n-ary function symbol, and $t_1, \ldots, t_n$ terms, then $ft_1 \ldots t_n$ is a term

Examples of terms

$(0, s, f, +, \cdot)$

0-ary

unary

binary

$f0sss0$, $+xfyz$, $\cdot + ab\, ss0$

$f(0, sss0)$ $\qquad$ $x + f(y, z)$ $\qquad$ $(a+b) \cdot ss0$

FIRST ORDER FORMULAS OVER $\mathcal{L}$

(1) $Pt_1 .. t_n$ is an atomic $\mathcal{L}$-formula, where
P is an n-ary predicate in $\mathcal{L}$, and
$t_1 .. t_n$ are terms over $\mathcal{L}$

(2) If A, B are $\mathcal{L}$-formulas, so are
$\neg A$, $(A \wedge B)$, $(A \vee B)$, $\forall x A$, $\exists x A$

Example: Propositional formulas are FO Formulas

$\mathcal{L}^{prop}$:

1. No function symbols
2. 0-ary predicate symbols $P_1, P_2, \ldots$ (are propositional atoms)

plus $\wedge, \vee, \neg,), ($

Since there are no function symbols, and all predicate symbols have 0-arity, propositional formulas have

No variables, terms, or $\forall, \exists$

Example : FO Formulas over $\mathcal{L}_A$

① Existence of infinitely many primes

$\forall x\ \exists y\ (y > x$ and y is prime$)$

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① Existence of infinitely many primes

Want to say: $\forall x\ \exists y\ (y > x$ and y is prime$)$

y is prime: $\forall z, z'\ (z, z' \geq 2 \Rightarrow z \cdot z' \neq y)$

Example: FO Formulas over $\mathcal{L}_A$

① Existence of infinitely many primes

want to say: $\forall x \; \exists y \; (\overbrace{y > x}^{(**)} \text{ and } \overbrace{y \text{ is prime}}^{(*)})$

y is prime : $\forall z, z' \; (z, z' \geq 2 \Rightarrow z \cdot z' \neq y)$

$$(*) \quad \forall z \forall z' \Big(\big(\neg(z=0) \wedge \neg(z=s0) \wedge \neg(z'=0) \wedge \neg(z'=s0) \big) \rightarrow \neg(z \cdot z' = y) \Big)$$

$A \rightarrow B$ abbreviates $\neg A \vee B$

Example: FO Formulas over $\mathcal{L}_A$

(1) Existence of infinitely many primes

Want to say: $\forall x \, \exists y \, (\underbrace{y > x}^{(**)} \text{ and } \overbrace{y \text{ is prime}}^{(*)})$

y is prime: $\forall z, z' \, (z, z' \geq 2 \Rightarrow z \cdot z' \neq y)$

(*) $\left[\forall z \forall z' \Big(\big(\neg(z=0) \wedge \neg(z = s0) \wedge \neg(z'=0) \wedge \neg(z' = s0) \big) \rightarrow \neg(z \cdot z' = y) \Big) \right.$

(**) $\left[y > x : \quad \neg(x = y) \wedge \exists w \, (x + w = y) \right.$

Example: FO Formulas over $\mathcal{L}_A$

① Existence of infinitely many primes

Want to express: $\forall x \exists y \, (\underbrace{y \text{ is prime}}_{A} \text{ and } \underbrace{y > x}_{B})$

A: $\forall z, z' \, (z, z' \geq 2 \Rightarrow z \cdot z' \neq y)$

$$\forall z \forall z' \left(\neg(z=0) \wedge \neg(z=S0) \wedge \neg(z'=0) \wedge \neg(z'=S0) \right) \to \neg(z \cdot z' = y)) \quad (*)$$

B: $\neg(x=y) \wedge \exists w \, (x+w=y) \qquad (**)$

Whole thing: $\forall x \exists y \, (*) \wedge (**)$

Example: FO Formulas over $\mathcal{L}_A$

(2) Twin Prime Conjecture

There exists infinitely many pairs of numbers, (x, x') such that $x' = x + 2$ and both x and x' are prime

Example : FO Formulas in $\mathcal{L}_A$

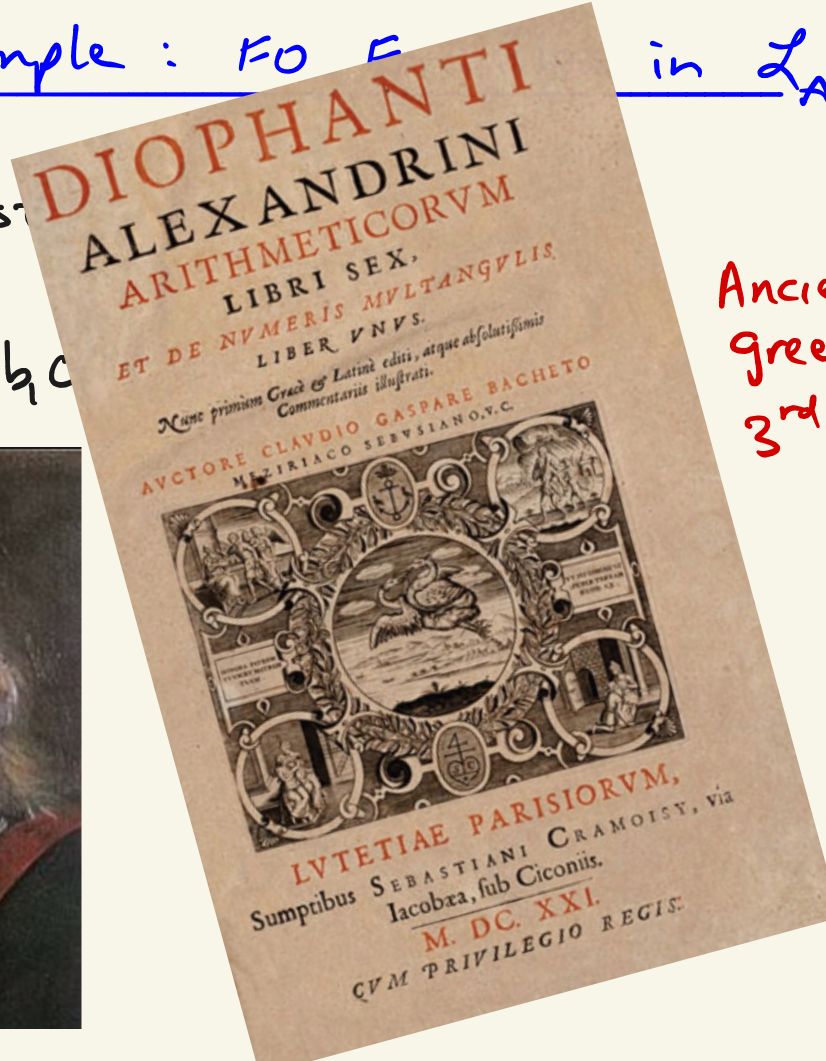
③ Fermat's Last Theorem

$$\forall n \geq 3 \; \forall a,b,c \; (n>2 \rightarrow a^n + b^n \neq c^n)$$

Example : FO F in $\mathcal{L}_A$

③ Fermat's Last

$\forall n \geq 3\ \forall a, b, c$



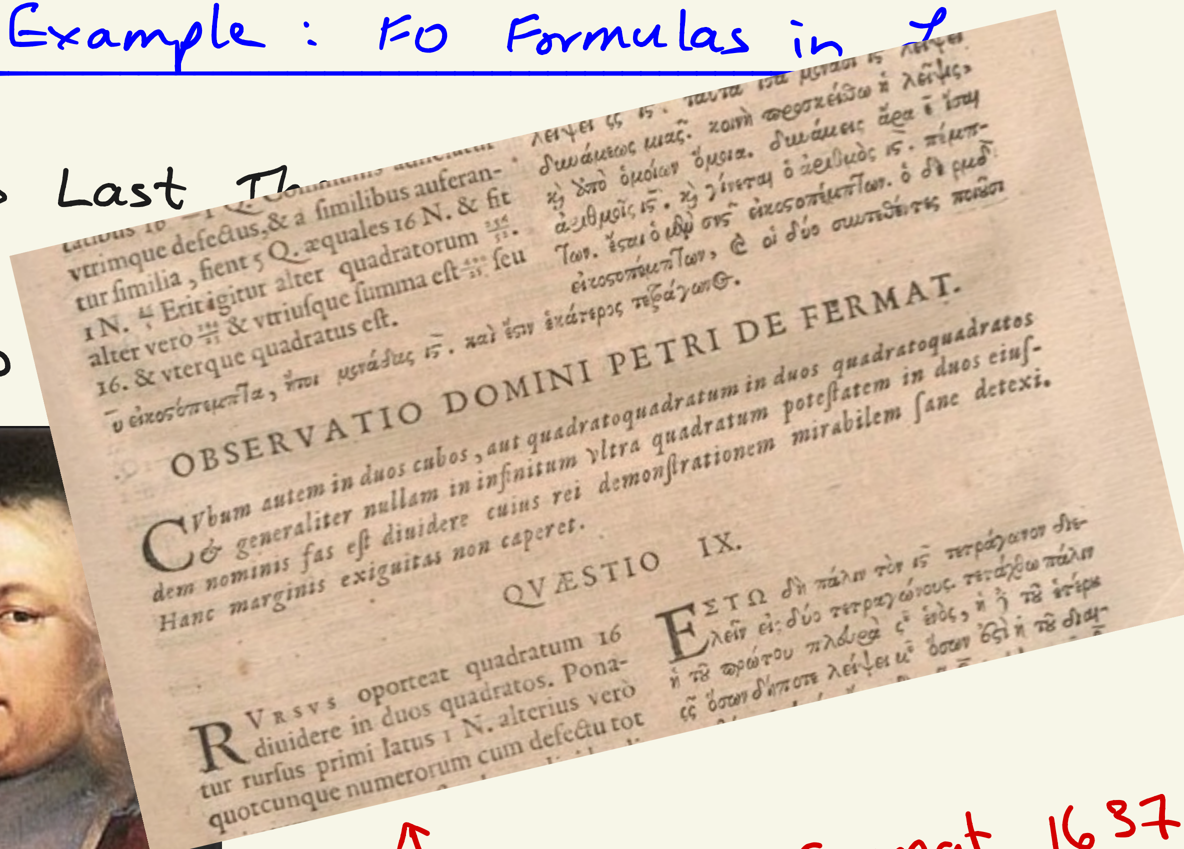
DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutissimis
Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V.C.
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via
Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Ancient greek text, 3rd century AD

Example : FO Formulas in $\mathcal{L}$

(3) Fermat's Last Th

$\forall n \geq 3$



latibus 16 — 1 Q. Communis adiiciatur
vtrimque defectus, & a similibus auferan-
tur similia, fient 5 Q. æquales 16 N. & fit
1 N. $\frac{16}{5}$ Erit igitur alter quadratorum $\frac{256}{25}$.
alter vero $\frac{144}{25}$ & vtriusque summa est $\frac{400}{25}$ seu
16. & vterque quadratus est.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

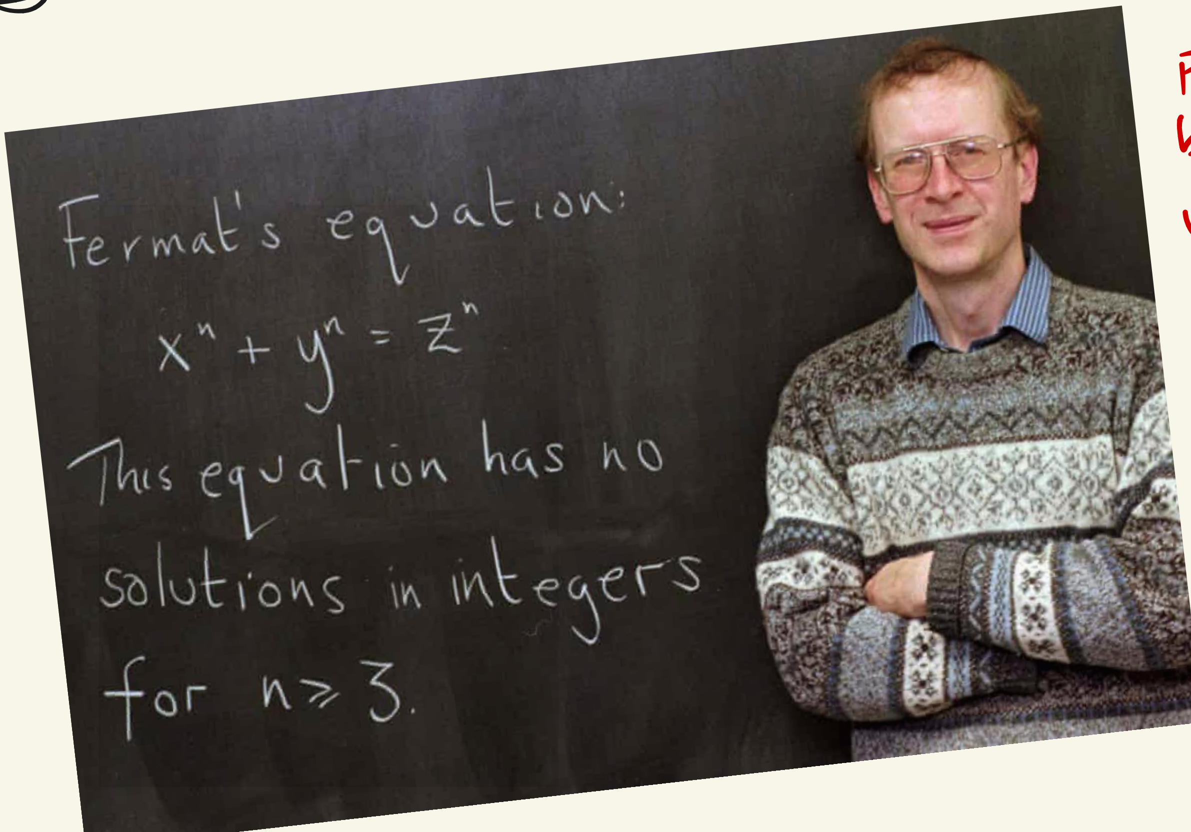
QVÆSTIO IX.

RVRSVS oporteat quadratum 16 diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot

conjectured by Fermat 1637
in margin of his copy of
Arithmetica

Example : FO Formulas in $\mathcal{L}_A$

③ Fermat's Last Theorem



Finally proven by Andrew Wiles

Example : FO Formulas in $\mathcal{L}_A$

③ Fermat's Last Theorem (actually Andrew Wiles' theorem)

$$\forall n \geq 3 \ (\forall a, b, c \quad a^n + b^n \neq c^n)$$

Problem: How to say a^n ?

(we'll see later how to do this!)

FREE / BOUND VARIABLES

- An occurrence of x in A is bound if x is in a subformula of A of the form $\forall x B$, or $\exists x B$ (otherwise x is free in A)

<u>Example</u> $\exists y (x = y + y)$

$Px \wedge \forall x (\neg (x + Sx = x))$

- A formula/term is closed if it contains no free variables
- A closed formula is called a sentence

SEMANTICS OF FO LOGIC

An $\mathcal{L}$-structure $\mathcal{M}$ (or model) consists of:

1. A nonempty set M called the universe (variables range over M)

2. For every n-ary function symbol f in $\mathcal{L}$, an associated function $f^{\mathcal{M}} : M^n \to M$

3. For each n-ary relation symbol P in $\mathcal{L}$, an associated relation $P^{\mathcal{M}} \subseteq M^n$

* Equality predicate $=$ is always true equality relation on M.

$M = \mathbb{N}$ * $=^{\mathbb{N}} = \{(i, i) \mid i \in \mathbb{N}\}$

Example

$\mathcal{L}_A = \{\bar{0}, +, \cdot, S; =\}$

$x = 4$
$z = 2$

1. $\mathbb{N}$: standard model of $\mathcal{L}_A$

$M = \mathbb{N}$

$\bar{0} = 0 \in \mathbb{N}$

$+, \cdot, S$ are usual plus, times, successor functions

Jumping ahead a bit: Evaluation of a formula in $\mathbb{N}$

$$\forall x \forall z \left(\exists z' \left(\neg(z' = 0) \wedge z + z' = x \right) \rightarrow \exists z'' \left(Sz + z'' = x \right) \right)$$

Says: $\forall x \forall z$ if $x > z$ then x can be written as $z+1$ + (some other $z'' \in \mathbb{N}$)

$$\underbrace{\forall x}\ \underbrace{\exists x}\ \left(\qquad\qquad \right)$$

Example

$\mathcal{L}_A = \{0, s, +, \cdot \,; =\}$

① $\mathcal{M} = \underline{\mathbb{N}}$,

$0 = 0 \in \mathbb{N}$

s: successor. ie. $s(2)=3$, ...

$+$: plus. ie, $+(0,1)=1$, $+(2,3)=5$, etc

$\cdot$: times

② $M = \{$■, ●, ★$\}$ $0 =$ ■

$s($■$) =$ ●

$s($●$) =$ ■

$s($★$) =$ ★

+	■	●	★
■	●	●	★
●	●	●	★
★	★	★	★

•	■	●	★
■	■	★	●
●	■	■	★
★	★	★	■

How to evaluate formulas that contain free variables?

<u>Defn</u> An **object assignment** σ for a model $\mathfrak{M}$ is a mapping from variables to M

Definition: Evaluation of terms/formulas over $\mathfrak{M}, \sigma$

Let $\mathfrak{M}$ be an $\mathcal{L}$-structure,

σ an object assignment for $\mathfrak{M}$

Evaluation of terms over $\mathfrak{M}, \sigma$

(a) $x^{\mathfrak{M}}[\sigma]$ is $\sigma(x)$ for all variables x

(b) $(f t_1 .. t_n)^{\mathfrak{M}}[\sigma] = f^{\mathfrak{M}}(t_1^{\mathfrak{M}}[\sigma], \ldots, t_n^{\mathfrak{M}}[\sigma])$

Example σ: $x_1 \to 5$ $x_2 \to 7$

$S(x_1 + x_2)[\sigma] = 13$

Evaluation of formulas over $\mathfrak{M}, \sigma$

Let A be an $\mathcal{L}$-formula. $\mathfrak{M} \models A[\sigma]$

($\mathfrak{M}$ satisfies A under σ) iff

(a) $\mathfrak{M} \models Pt_1 \ldots t_n[\sigma]$ iff $\langle t_1^{\mathfrak{M}}[\sigma], \ldots, t_n^{\mathfrak{M}}[\sigma] \rangle \in P^{\mathfrak{M}}$

(b) $\mathfrak{M} \models (s = t)[\sigma]$ iff $s^{\mathfrak{M}}[\sigma] = t^{\mathfrak{M}}[\sigma]$

(c) $\mathfrak{M} \models \neg A[\sigma]$ iff not $\mathfrak{M} \models A[\sigma]$

(d) $\mathfrak{M} \models (A \vee B)[\sigma]$ iff $\mathfrak{M} \models A[\sigma]$ or $\mathfrak{M} \models B[\sigma]$

(e) $\mathfrak{M} \models (A \wedge B)[\sigma]$ iff $\mathfrak{M} \models A[\sigma]$ and $\mathfrak{M} \models B(\sigma)$

(f) $\mathfrak{M} \models \forall x A[\sigma]$ iff $\forall m \in M$ $\mathfrak{M} \models A[\sigma(m/x)]$

(g) $\mathfrak{M} \models \exists x A[\sigma]$ iff $\exists m \in M$ $\mathfrak{M} \models A[\sigma(m/x)]$

A

$$\forall x \, \exists x \, (x + x = SS0 + x)$$

$$\iff \forall x \, (x + x = SS0 + x)$$

For $a = 0, 1, 2, \ldots$

Evaluate $\exists x \, (x + x = SS0 + x)[a/x]$

For $b = 0, 1, 2 \ldots$

Evaluate $(x + x + SS0 + x)[b/x][a/x]$

Example

$\mathcal{L} = \{\,;\, R, =\}$

$\mathfrak{M} = (\mathbb{N}, \leq, =)$

$R^{\mathfrak{M}}(m,n)$ iff $m \leq n$

Then $\mathfrak{M} \overset{?}{\models} \forall x \exists y\, R(x,y)$ yes

satisfiable by $\mathfrak{M}$

$\mathfrak{M} \overset{?}{\models} \exists y \forall x\, R(x,y)$ No

but $\exists y \forall x R(x,y)$ is also satisfiable

IMPORTANT DEFINITIONS

Let A be a f.o. formula over $\mathcal{L}$.

① A is satisfiable iff there exists a model $\mathfrak{M}$ and an object assignment σ such that $\mathfrak{M} \models A[\sigma]$

② A set of formulas Φ is satisfiable iff $\exists \mathfrak{M}, \sigma$ such that $\mathfrak{M} \models \Phi[\sigma]$ $\left[\mathfrak{M} \models A[\sigma] \text{ for all } A \in \Phi\right]$

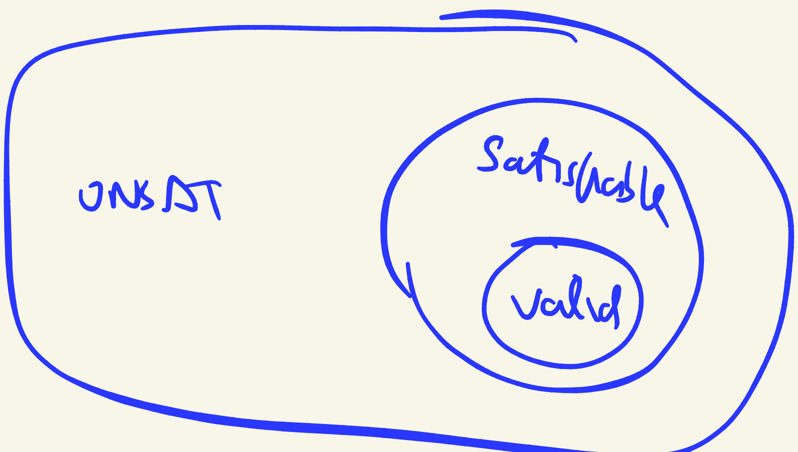
③ $\Phi \models A$ (A is a logical consequence of Φ) iff $\forall \mathfrak{M}\ \forall \sigma$ if $\mathfrak{M} \models \Phi[\sigma]$ then $\mathfrak{M} \models A[\sigma]$

$\models A$ (A is valid) iff $\forall \mathfrak{M}, \sigma$ $\mathfrak{M} \models A[\sigma]$

④ $A \Leftrightarrow B$ (A and B are logically equivalent)

iff $\forall \mathfrak{M}\ \forall \sigma \quad \mathfrak{M} \models A[\sigma]$ iff $\mathfrak{M} \models B[\sigma]$

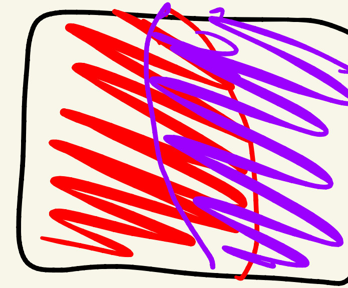
$A \models B$ and $B \models A$



Examples

The

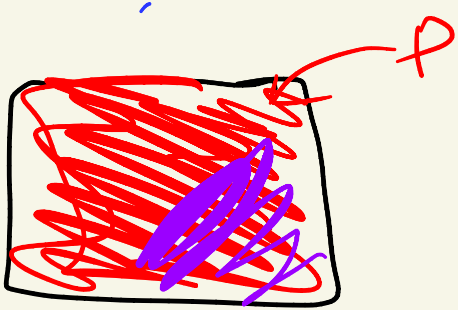
① $(\forall x\, Px \lor \forall x . Qx) \overset{?}{\models} \forall x (Px \lor Qx)$



② $\forall x (Ax \lor Bx) \overset{?}{\models} \forall x\, Ax \lor \forall x\, Bx$

$$\mathcal{L} = \{ ; P, Q, A, B \}$$

M



Example

Earlier formula A:

$$\forall x \forall z \; (\exists z' (\neg(z' = 0) \wedge z + z' = x) \supset$$
$$\exists z'' (sz + z'' = x))$$

says for every x, z if $x > z$ then
we can write x as $(z+1) + z''$ for some z''

- true when $\mathcal{M} = \underline{\mathbb{N}}$ so A is satisfiable
- false when $\mathcal{M} = \Big(M = \{0,1,2\}$ $\quad s0 = 1$, $s1 = 2$, $s2 = 0$ $\quad 0 + 2 = 2$, all others $x + y = 0 \Big)$

$x = 2 \quad z = 0$

Example

$$\forall x \forall y \, (fx = fy) \overset{?}{\models} x = y$$

Construct model $\mathcal{M}$, and σ

$M = \{0, 1, 2\}$

$f(x) = 0 \quad \forall x \in \{0, 1, 2\}$

$\forall x \forall y \; f(x) = f(y) = 0$

But

$\sigma(x) = 1$

$\sigma(y) = 2$

Example

$$\forall x \forall y\,(fx = fy) \overset{?}{\models} x = y$$

No

Let $M = \{0, 1\}$

$\mathfrak{M}$: $f(0) = 0$

$f(1) = 0$

then $\mathfrak{M} \models \forall x \forall y\,(fx = fy)$

but $\mathfrak{M} \not\models x = y$ (since $0 \neq 1$)

Substitution

Let s,t be $\mathcal{L}$-terms.

$t(s/x)$: substitute x everywhere by s

$A(s/x)$: substitute all free occurrences of x in A by s

For readability, we will write

$t = +SS0\,x$ as $SS0 + x$

Substitution

Let s, t be $\mathcal{L}$-terms.

$t(s/x)$: substitute x everywhere by s

$A(s/x)$: substitute all free occurrences of x in A by s

Lemma $$(t(s/x))^{\mathcal{M}}[\sigma] = t^{\mathcal{M}}\left[\sigma\left(\frac{s^{\mathcal{M}}[\sigma]}{x}\right)\right]$$

substitute x for s to get t' then evaluate t' under $\mathcal{M}, \sigma$

obtain new object assignment σ' where $\sigma'(x) = s^{\mathcal{M}}$ then evaluate t under $\mathcal{M}, \sigma'$

Substitution Cont'd

Need to be more careful when making substitutions into formulas

Example: $A : \forall y \neg (x = y + y)$

$$A\left(\frac{x+y}{x}\right) : \forall y \neg (x + y = y + y)$$

Defn term t is freely substitutable for x in A iff there is no subformula in A of the form $\forall y B$ or $\exists y B$ where y occurs in t

Substitution Theorem

If t is freely substitutable for x in A

then $\forall \mathfrak{M} \ \forall \sigma$

$$\mathfrak{M} \models A(t/x)[\sigma] \quad \text{iff} \quad \mathfrak{M} \models A[\sigma(t^{\mathfrak{M}}[\sigma]/x)]$$

Easy way to avoid this problem
(of making a "bad" substitution):

2 types of variables
free variables $a, b, c, \ldots$

bound variables $x, y, z, \ldots$

Proper formula: every free variable occurrence is of type free & every bound variable occurrence of type bound

Proper term: No variables of type bound

FIRST ORDER SEQUENT CALCULUS LK

Lines are again sequents

$$\left.A_1, \ldots, A_k \rightarrow B_1, \ldots, B_\ell \right\} S$$

where each A_i, B_j is a proper $\mathcal{L}$-formula

$$A_S : A_1 \wedge A_2 \wedge \ldots \wedge A_k \supset B_1 \vee \ldots \vee B_\ell$$

FIRST ORDER SEQUENT CALCULUS LK

Lines are again sequents

$$A_1, \ldots, A_k \rightarrow B_1, \ldots, B_\ell$$

where each A_i, B_j is a proper $\mathcal{L}$-formula

RULES

OLD RULES OF PK

PLUS NEW RULES FOR $\forall$, $\exists$

like a large AND

Large OR

New Logical Rules for $\forall$, $\exists$

$\forall$-left $\quad \dfrac{A(t), \Gamma \to \Delta}{\forall x\, A(x), \Gamma \to \Delta}$ $\qquad$ $\forall$-Right $\quad \dfrac{\Gamma \to \Delta, A(b)}{\Gamma \to \Delta, \forall x\, A(x)}$

$\exists$-left $\quad \dfrac{A(b), \Gamma \to \Delta}{\exists x\, A(x), \Gamma \to \Delta}$ $\qquad$ $\exists$-right $\quad \dfrac{\Gamma \to \Delta, A(t)}{\Gamma \to \Delta, \exists x\, A(x)}$

* A, t are proper

* b is a free variable Not appearing in lower sequent of rule

Example of an LK proof

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{Pa \to Pa}{Pa, Qa \to Pa}}{Pa \wedge Qa \to Pa}\,\text{AND-left}}{Pa \wedge Qa \to \exists x Px}\,\exists\text{-Rt}}{\exists x(Px \wedge Qx) \to \exists x Px}\,\exists\text{ Left}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{Qa \to Qa}{Pa, Qa \to Qa}}{Pa \wedge Qa \to Qa}\,\text{AND-Left}}{Pa \wedge Qa \to \exists x Qx}\,\exists\text{-Rt}}{\exists x(Px \wedge Qx) \to \exists x Qx}\,\exists\text{-Left}
}{\exists x(Px \wedge Qx) \to \exists x Px \wedge \exists x Qx}\,\text{AND-RT}
$$

SOUNDNESS

Defn A first order sequent $A_1, \ldots, A_k \rightarrow B_1, \ldots, B_\ell$ is valid if and only if its associated formula $(A_1 \wedge \ldots \wedge A_k) \supset (B_1 \vee \ldots \vee B_\ell)$ is valid.

Soundness Theorem for LK Every sequent provable in LK is valid

Proof of Lemma

go through each rule

Example: $\forall$-right rule

$$\frac{\Gamma \to \Delta, A(a)}{\Gamma \to \Delta, \forall x\, A(x)} \quad \begin{matrix} \leftarrow S \\ \leftarrow S' \end{matrix}$$

Note: a cannot occur in lower sequent & thus a cant occur in any B_i, C_j

Let $\Gamma = B_1 \dots B_\ell$
$\Delta = C_1 \dots C_{\ell'}$

A_S : $B_1 \wedge \dots \wedge B_\ell \supset C_1 \vee \dots \vee C_{\ell'} \vee A(a)$

$A_{S'}$: $B_1 \wedge \dots \wedge B_\ell \supset C_1 \vee \dots \vee C_{\ell'} \vee \forall A(x)$

Theorem (LK Soundness)

Every sequent provable in LK is valid

Pf by induction on the number of sequents in proof.

Axiom $A \to A$ is valid

Induction step : use previous soundness lemma

Soundness (Proof) : By induction on the number of sequents in proof

Example: $\exists$ Left

Assume: $A(b), \Gamma \Rightarrow \Delta$ has an LK proof and is valid

show: $\exists x\, A(x), \Gamma \Rightarrow \Delta$ also valid

By defn $\overline{A(b)} \vee \overline{\Gamma}_1 \vee .. \vee \overline{\Gamma}_k \vee \Delta_1 \vee .. \vee \Delta_k$ is valid

Let $\mathfrak{M}$ be any structure, σ any object assignment.

Show: $\mathfrak{M} \models \neg \exists x\, A(x) \vee \overline{\Gamma}_1 \vee .. \vee \overline{\Gamma}_k \vee \Delta_1 \vee .. \vee \Delta_k\ [\sigma]$ (*)

Case 1 $\mathfrak{M} \models \overline{\Gamma}_1 \vee .. \vee \overline{\Gamma}_k \vee \Delta_1 \vee ... \vee \Delta_k\ [\sigma]$. Then (*) holds

Case 2 Case 1 does not hold.

Soundness (Proof): By induction on the number of sequents in proof

Example: $\exists$ Left

Assume: $A(b), \Gamma \Rightarrow \Delta$ has an LK proof and is valid

Show: $\exists x A(x), \Gamma \Rightarrow \Delta$ also valid

By defn $\overline{A(b)} \vee \overline{\Gamma}_1 \vee \ldots \vee \overline{\Gamma}_k \vee \Delta_1 \vee \ldots \vee \Delta_k$ is valid

Let $\mathfrak{M}$ be any structure, σ any object assignment.

Show: $\mathfrak{M} \models \neg \exists x A(x) \vee \overline{\Gamma}_1 \vee \ldots \vee \overline{\Gamma}_k \vee \Delta_1 \vee \ldots \vee \Delta_k \; [\sigma]$ $(*)$

Case 1 $\quad \mathfrak{M} \models \overline{\Gamma}_1 \vee \ldots \vee \overline{\Gamma}_k \vee \Delta_1 \vee \ldots \vee \Delta_k \; [\sigma]$. Then $(*)$ holds

Case 2 $\quad$ Case 1 does not hold.

Since b does not occur in Γ or Δ,

$$\mathfrak{M} \not\models \overline{\Gamma}_1 \vee \ldots \vee \overline{\Gamma}_k \vee \Delta_1 \vee \ldots \vee \Delta_k \; [\sigma(m/b)] \quad \text{for all } m \in M$$

Since $A(b), \Gamma \Rightarrow \Delta$ is valid, $\mathfrak{M} \models \overline{A(b)} \; [\sigma[m/b]] \; \forall m \in M$

Thus $M \models \neg \exists x A(x) \; [\sigma]$, & thus $\exists x A(x), \Gamma \Rightarrow \Delta$ is valid