Last Class:

1. Intro
2. Propositional logic

   Syntax / semantics

   Resolution: proof system for propositional logic

   - soundness
   - completeness

Pages 1-9 of Lecture Notes, plus supplementary notes on Resolution

Announcements

- Office hours start next week

Monday 5-6 (Toni, via zoom)

Tues 2:45 - 3:45 (Yasaman, in person 5th floor Mudd)

Last Class (Recap)

- Course syllabus, overview/intro
- Propositional Logic
- Resolution Proof System
    - Soundness
    - Completeness

Welcome to CS 4995 : Computability and Logic

Professor : Toniann Pitassi (Toni)

TA : Yasaman Mahdaviyeh

Webpage :

www.cs.columbia.edu/~toni/courses/Logic2022/4995.html

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* Email ahead of time if you plan to attend *

Marking Scheme

2 assignments	20% each
2 tests (in class)	25% each
class participation	10%

Dates

Homework 1	Oct 11	11:59 pm
Test 1	Oct 19	in class
Homework 2	Nov 29	11:59 pm
Test 2	Dec 7	in class

PROPOSITIONAL LOGIC

Inductive Definition of a Propositional Formula

1. Atoms/Propositional variables: $P_1, P_2, \ldots x, y, z, \ldots$ are formulas

2. If A is a formula, then so is $\neg A$
3. If A, B are formulas, so is $(A \wedge B)$
4. " " " " " " $(A \vee B)$

Example: $((x \vee y) \wedge (\neg x)) \vee (z \vee \neg y)$

Semantics

A truth assignment $\tau : \{\overset{\text{propvariable}}{\text{atoms}}\} \to \overset{\text{true}}{T}, \overset{\text{false}}{F}$

Extending τ to every formula:

(1) $(\neg A)^\tau = T$ iff $A^\tau = F$

(2) $(A \wedge B)^\tau = T$ iff $A^\tau = T \wedge B^\tau = T$

(3) $(A \vee B)^\tau = T$ iff either $A^\tau = T$ or $B^\tau = T$

Definitions.

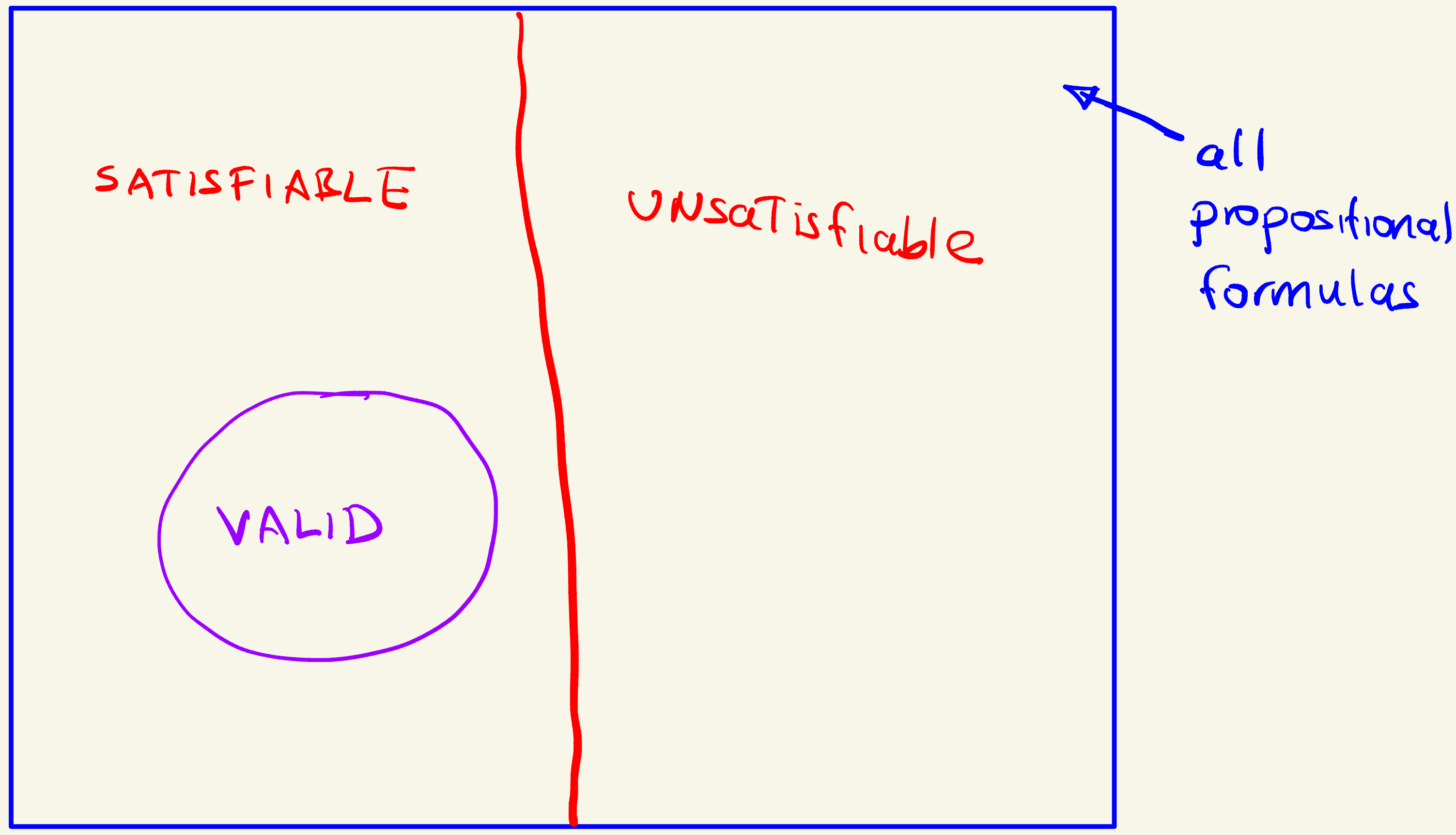
τ satisfies A iff $A^{\tau} = T$

A is satisfiable iff there exists some truth assignment τ such that $A^{\tau} = T$

A is unsatisfiable iff for every truth assignment τ, $A^{\tau} = F$

A is a tautology (or valid) iff for every truth assignment τ, $A^{\tau} = T$

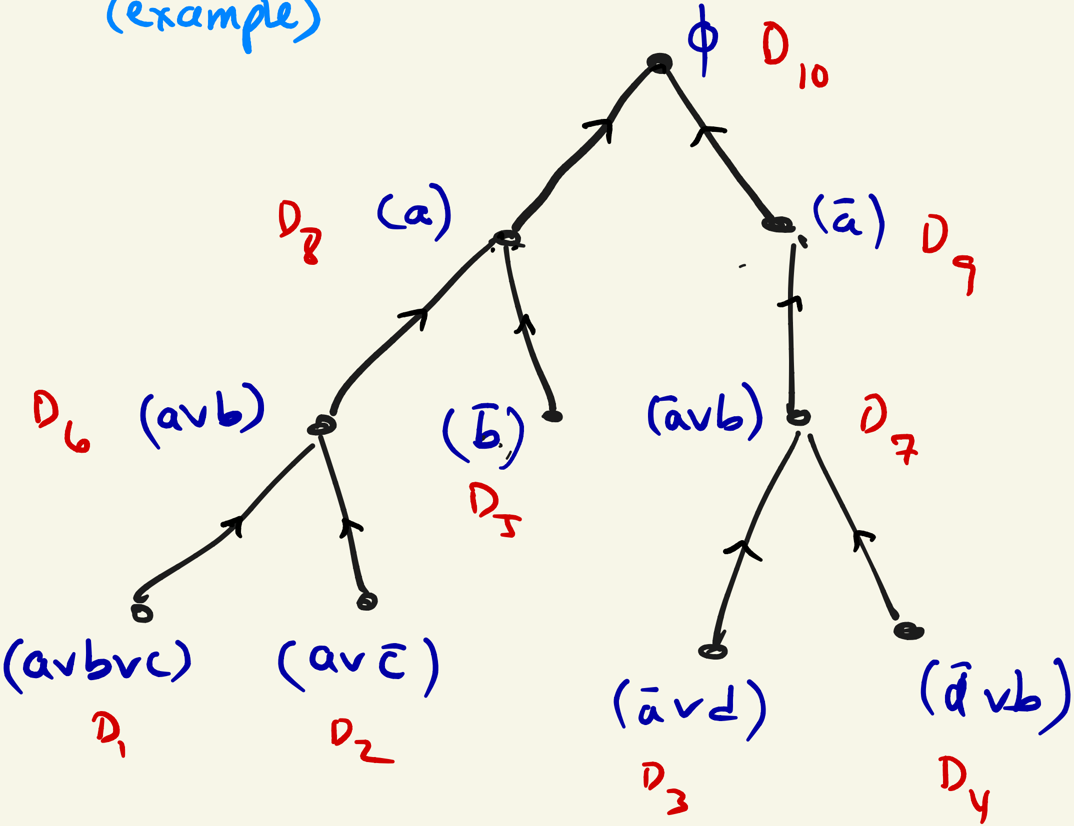
$\Phi \models A$ $\forall \tau$ (If τ satisfies Φ then τ must satisfy A)


SATISFIABLE
unsatisfiable
VALID
all propositional formulas

Resolution Refutation

(example)

$F = (a \vee b \vee c)(a \vee \bar{c})(\bar{b})(\bar{a} \vee d)(\bar{d} \vee b)$



Res Rule

$(A \vee x) \quad (B \vee \bar{x})$

$(A \vee B)$

Resolution Soundness

<u>Fact</u>: If C_1, C_2 derive C_3 by Resolution rule, then $C_1, C_2 \models C_3$

(Exercise: Show $(A \vee x), (B \vee \overline{x}) \models (A \vee B)$)

From above Fact we can prove by induction:

RESOLUTION SOUNDNESS THEOREM

If a CNF formula F has a RES refutation, then F is unsatisfiable

RESOLUTION COMPLETENESS THM

Every unsatisfiable CNF formula F has a RESOLUTION Refutation

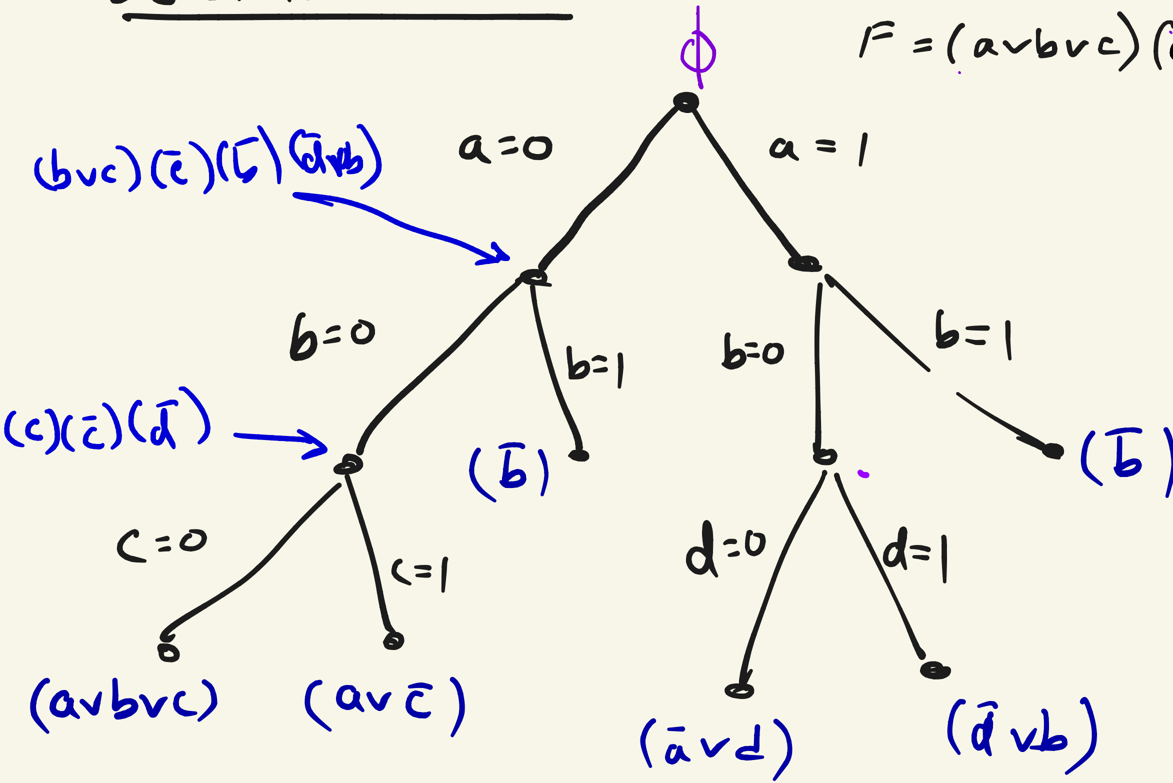
Proof idea

We describe a canonical procedure for obtaining a RES refutation for F

The procedure exhaustively tries all truth ass's – via a decision tree
then we show that any such decision tree can be viewed as a RES refutation.

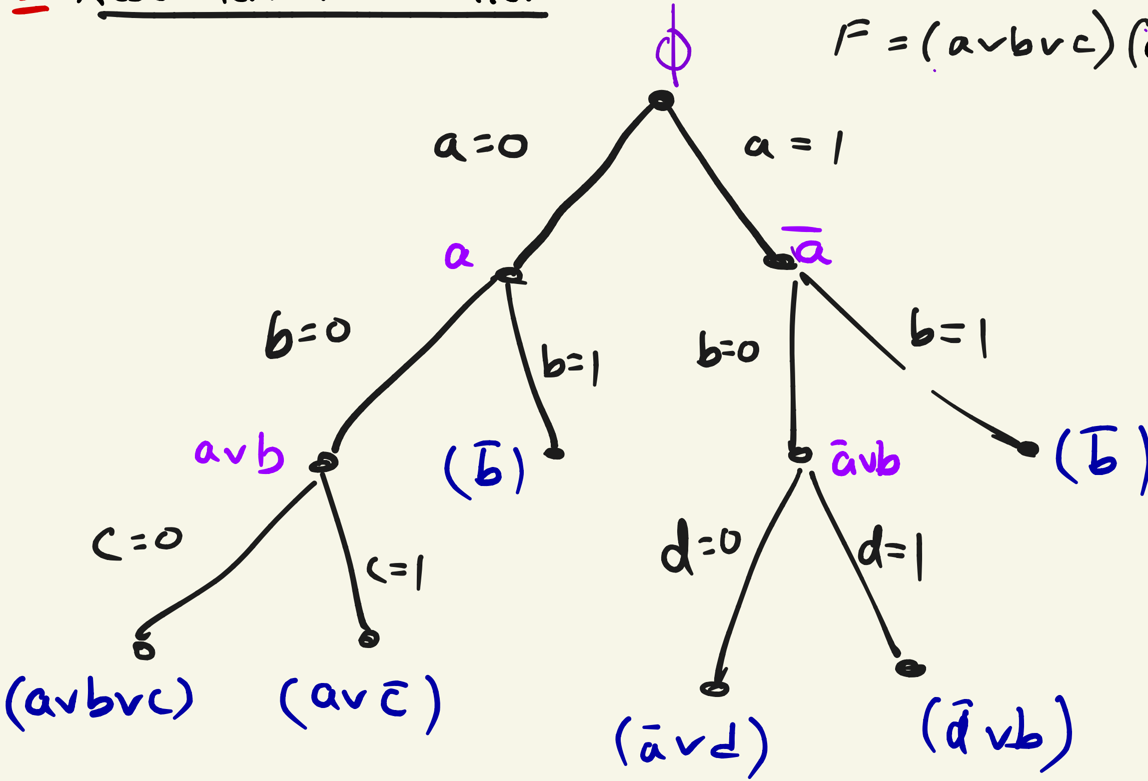
DECISION TREES

$$F = (a \vee b \vee c)(a \vee \bar{c})(\bar{b})(\bar{a} \vee d)(\bar{d} \vee b)$$



Resolution Refutation

$F = (a \vee b \vee c)(a \vee \bar{c})(\bar{b})(\bar{a} \vee d)(\bar{d} \vee b)$



Today

- Another proof system for propositional logic: PK
  - soundness of PK
  - completeness of PK
- Derivational Soundness/Completeness of PK
- Propositional compactness Theorem

Pages 9 – 17 of Lecture Notes

THE (PROPOSITIONAL) SEQUENT CALCULUS

- A second sound + complete system for propositional logic

- More natural for proving theorems (but more difficult as basis for automated theorem proving / SAT solving)

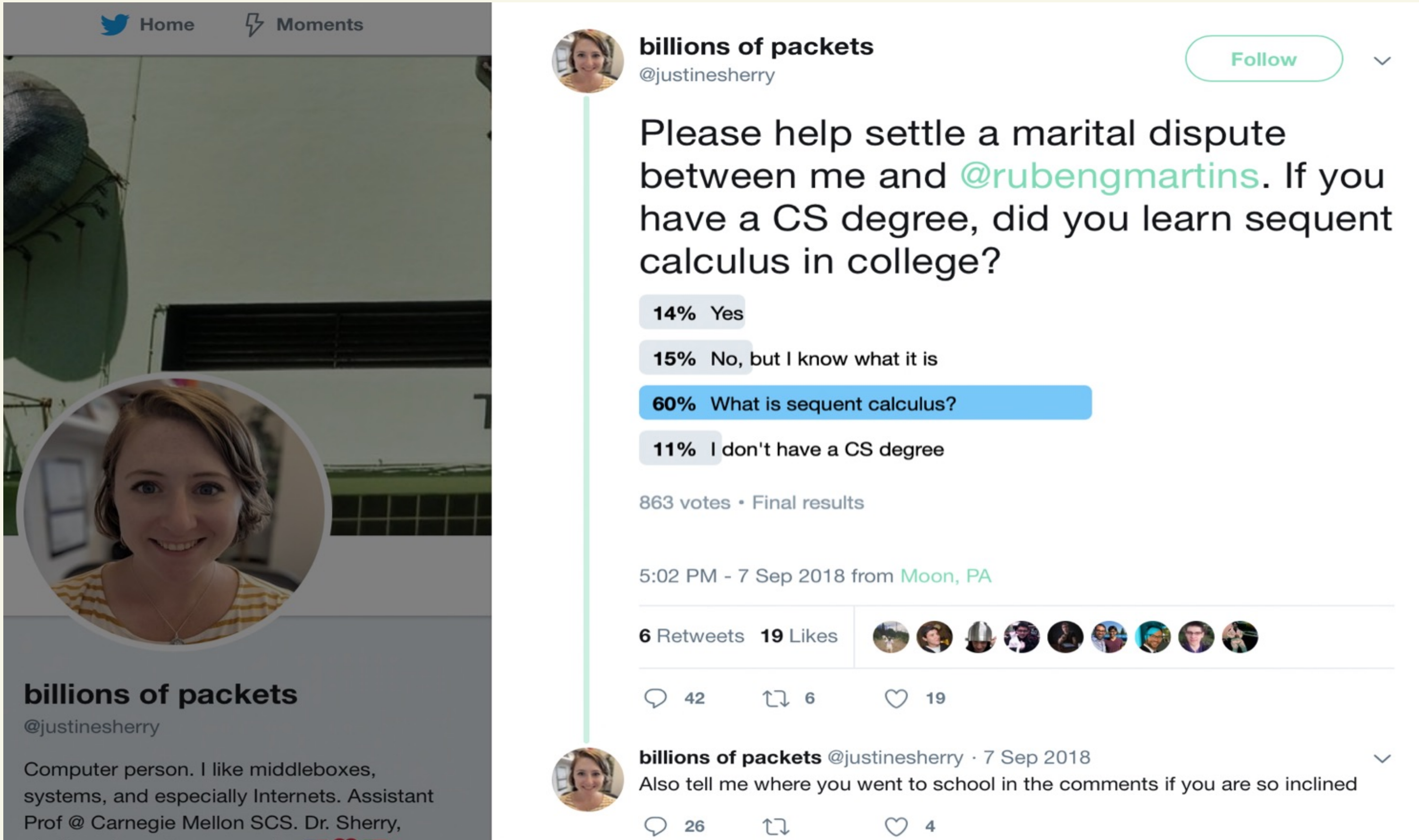
Why I love the Sequent Calculus

- Rules are very natural – 2 for each boolean connective

- Cut Rule (Modus Ponens) Not needed for completeness. (This formulation requires meta-symbol "$\rightarrow$")

  This greatly simplifies / clarifies completeness proofs for propositional and first order logic

Sequent Calculus goes viral on Twitter



Home Moments

4 17

Pierce Darragh 🍴 @_pdarragh · 7 Sep 2018

Pretty sure it's the calculus of how to put shiny plastic things on clothing

1 8

John Regehr @johnregehr · 7 Sep 2018

YES!!!

2

kat 猫 ✨🐙✨ @wirehead2501 · 7 Sep 2018

Replying to @justinesherry @rubengmartins

@chrisamaphone no; i have a CS BS and comp eng MS and only had to take Calc I and II. don't think i've ever heard the term "sequent"

2

Chris Martens @chrisamaphone · 8 Sep 2018

But surely you have heard its plural, as in, “a sequents of events” 😬

1 6

jason reed @jcreed · 8 Sep 2018

if you are against bedazzling: sequin't

1 3

jason reed @jcreed · 8 Sep 2018

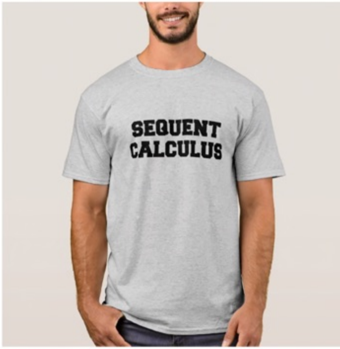
oh wait I have confused sequins with rhinestones, my bad

3

ions of packets

tinesherry

puter person. I like middleboxes, ems, and especially Internets. Assistant @ Carnegie Mellon SCS. Dr. Sherry,



Sequent Calculus Standard Shirt

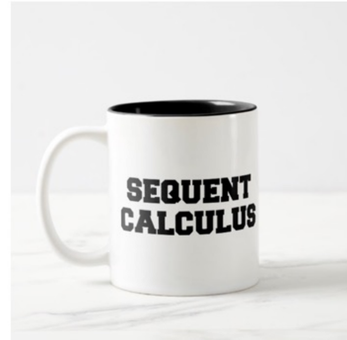
$18.95

15% Off with code GANGSALLHERE

Sequent Calculus Fitted Shirt

$17.95

15% Off with code GANGSALLHERE



Sequent Calculus Mug

$16.95

15% Off with code GANGSALLHERE

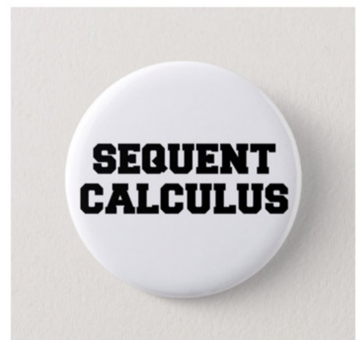
Sequent Calculus Tote Bag

$9.95

15% Off with code GANGSALLHERE

German Fitted Sequent Calculus Shirt

German Sequent Calculus Shirt



Sequent Calculus Pin

Gentzen's PK proof system

Lines in a PK proof are **sequents**

$$\underbrace{A_1, \ldots, A_k}_{\text{antecedent}} \rightarrow \underbrace{B_1, \ldots, B_r}_{\text{succedent}}$$

$A_1, \ldots, A_k, B_1, \ldots, B_r$ are propositional formulas

$\rightarrow$ is a <u>new</u> symbol (NOT part of language of propositional logic)

Gentzen's PK proof system

Lines in a PK proof are sequents

$$A_1, \ldots, A_k \longrightarrow B_1, \ldots, B_r$$

Semantics:

$$A_1 \wedge A_2 \wedge \ldots \wedge A_k \supset B_1 \vee \ldots \vee B_r$$

the conjunction of the A_i's implies the disjunction of the B_i's

gentzen's PK proof system

Lines in a PK proof are **sequents**

$$A_1, \dots, A_k \rightarrow B_1, \dots, B_r \quad \stackrel{d}{=} \; S$$

Semantics:

$$A_1 \wedge A_2 \wedge \dots \wedge A_k \supset B_1 \vee \dots \vee B_r \quad \stackrel{d}{=} \; A_S$$

the conjunction of the A_i's implies the disjunction of the B_i's

$$S \stackrel{d}{=} A_1, \ldots, A_k \to B_1, \ldots, B_j$$ sequent

$$A_S : \neg(A_1 \wedge \ldots \wedge A_k) \vee (B_1 \vee \ldots \vee B_j)$$ associated propositional formula

Gentzen's PK proof system

Lines in a PK proof are **sequents**

$S \stackrel{d}{=} A_1, \ldots, A_k \longrightarrow B_1, \ldots, B_q$ $\longleftarrow A_S \doteq \neg(A_1 \wedge \ldots \wedge A_k)$

0 or False

1 or True

Convention: Empty conjunction (antecedent empty) $\to 1$

Empty disjunction (succedent empty) $\longrightarrow 0$

PK Rules

Intuitively: Structural Rules (cedents are sets)

Logical Rules (define the boolean connectives $\wedge, \vee, \neg$)

Cut Rule

STRUCTURAL RULES

	Left	Right
Weakening	$\dfrac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}$	$\dfrac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}$
Exchange	$\dfrac{\Gamma_1, A, B, \Gamma_2 \rightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \rightarrow \Delta}$	$\dfrac{\Gamma \rightarrow \Delta_1, A, B, \Delta_2}{\Gamma \rightarrow \Delta_1, B, A, \Delta_2}$
Contraction	$\dfrac{\Gamma, A, A \rightarrow \Delta}{\Gamma, A \rightarrow \Delta}$	$\dfrac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A}$

LOGICAL RULES

	Left	Right
$\neg$ Intro	$\dfrac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$	$\dfrac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$
$\wedge$ Intro	$\dfrac{A, B, \Gamma \Rightarrow \Delta}{(A \wedge B), \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, (A \wedge B)}$
$\vee$ Intro	$\dfrac{A, \Gamma \Rightarrow \Delta, \quad B, \Gamma \Rightarrow \Delta}{(A \vee B), \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, (A \vee B)}$

So we can think of

$$A_1, \dots, A_k \rightarrow B_1, \dots B_j$$

as $\rightarrow \neg A_1 \vee \neg A_2 \vee \neg A_3 \vee \dots \vee \neg A_k \vee B_1 \vee \dots \vee B_j$

CUT RULE

$$\frac{\Gamma \to \Delta, A \qquad \Gamma, A \to \Delta}{\Gamma \to \Delta}$$

Axiom

$$A \to A \qquad \equiv \qquad \underbrace{\neg A \vee A}_{\text{law of excluded middle}}$$

Example: A PK proof of a formula A is a PK proof of $\to A$

$$\dfrac{P \to P}{P \to Q, P} \qquad \dfrac{Q \to Q}{Q \to P, \cancel{Q}} \qquad \text{weakening}$$

$\leftarrow$ ¬ Left

$$\dfrac{P \to Q, P}{P, \neg Q \to P} \qquad \dfrac{Q \to P, Q}{Q, \neg P \to Q} \qquad \neg\text{-Left}$$

$$\dfrac{P, \neg Q \to P}{P, \neg P, \neg Q \to} \qquad \dfrac{Q, \neg P \to Q}{Q, \neg P, \neg Q \to} \qquad \neg\text{-Left}$$

$$\dfrac{P, \neg P, \neg Q \to \qquad Q, \neg P, \neg Q \to}{(P \vee Q), \neg P, \neg Q \to} \qquad \text{OR LEFT}$$

$$\dfrac{(P \vee Q), \neg P, \neg Q \to}{(P \vee Q), \neg P \wedge \neg Q \to} \qquad \text{AND-Left}$$

$$\dfrac{(P \vee Q), \neg P \wedge \neg Q \to}{(\neg P \wedge \neg Q) \to \neg(P \vee Q)} \qquad \neg\ \text{Right}$$

PK SOUNDNESS : Every sequent provable in PK is VALID

As in the propositional case, we first verify the soundness of all rules + then prove PK soundness by induction

<u>Lemma</u> (Soundness of Rules)
For every rule of PK, if all top sequents are valid, then the bottom sequent is valid
also the axiom is valid

<u>PK soundness</u> If S has a PK proof, then A_S is valid

Example

$$\frac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, (A \wedge B)}$$

AND-rt rule

Let $\Gamma = C_1, \ldots, C_k$

$\Delta = D_1, \ldots, D_\ell$

Show:

Let τ be any truth ass to atoms in $\Gamma, \Delta, (A \vee B)$

If τ satisfies $\neg(C_1 \wedge \ldots \wedge C_k) \vee (D_1 \vee \ldots \vee D_\ell) \vee A$ — formula corresponding to $\Gamma \to \Delta, A$

and τ satisfies $\neg(C_1 \wedge \ldots \wedge C_k) \vee (D_1 \vee \ldots \vee D_\ell) \vee B$ — formula for $\Gamma \to \Delta, B$

Then τ satisfies $\neg(C_1 \wedge \ldots \wedge C_k) \vee (D_1 \vee \ldots \vee D_\ell) \vee (A \wedge B)$

PK COMPLETENESS: Every valid propositional sequent has a PK proof

- How to prove completeness?

  One idea is to bootstrap off of fact that Resolution is complete.

  Show: any Resolution refutation can be simulated in PK.

  Then since RES is complete (for refuting UNSAT CNFs), so is PK.

- We'll give a more direct proof, via an algorithm that constructs a PK proof of any f that is a tautology

PK COMPLETENESS: Every valid propositional sequent has a PK proof

Main idea: again we will give an algorithm that will produce a PK proof for any valid sequent

Algorithm: Write sequent at bottom (of proof)

1. Repeatedly: pick an outermost connective in a formula in a leaf sequent of current proof + apply the rule for that connective (in reverse)

2. Continue until all leaf sequents consist of just atoms

3. For each leaf sequent, ~~if~~ try to apply weakening in reverse to get $p \rightarrow p$

<u>Show</u>: If we run algorithm on a <u>valid</u> sequent $\Gamma \to \Delta$, then at end, all leaf sequents must contain an atom occurring both on left & right — ie $A, B, C \to A, D$

Then can finish proof by applying weakening (in reverse)

ie. $$\frac{A \to A}{A, B, C \to A, D}$$

PK completeness (cont'd)

Key Property is the INVERSION PRINCIPLE:

each PK rule except weakening has the property that $\forall$ truth assignments τ, if τ satisfies bottom sequent, then τ satisfies both upper sequents

* Called inversion since it is the reverse direction of what we needed to prove soundness: $\forall \tau$ if τ satisfies both upper sequents, then τ satisfies lower sequent

PK completeness

- If $\Gamma \to \Delta$ is valid, by Inversion Property, all leaf sequents generated in step 1 of Algorithm are VALID, and have one less connective than sequent below
- Thus eventually step 1 halts, where each leaf sequent involves only atoms and each leaf sequent is valid

  $\therefore$ each leaf sequent looks like $A, \Gamma \to A, \Delta$

  ie has an atom A on both Left & Rt sides

PK completeness

Claim Let π be the output of PK algorithm on input $\Gamma \to \Delta$.
If $\Gamma \to \Delta$ is valid then every leaf sequent of π contains only atomic formulas (propositional variables) on left/right of "$\to$" and furthermore there exists some atomic formula occurring on both the left + right

Pf Suppose for sake of contradiction that some leaf sequent of π is: $x_1, .., x_k \to y_1, .., y_\ell$
where $\{x_1, .., x_k\} \cap \{y_1 .. y_\ell\} = \phi$. Then the truth assignment τ that sets $x_1 = x_2 = .. = x_k = 1$, $y_1 = y_2 = .. = y_\ell = 0$ falsifies the sequent which contradicts fact that all leaf sequents of π are valid (since $\Gamma \to \Delta$ assumed to be valid)

Cut-Elimination Theorem for PK

If $\Gamma \to \Delta$ has a PK proof, then it has a proof with no use of the cut rule.

Derivational Soundness & Completeness of PK

Definition Let Φ be a set of sequents, S a sequent

A PK-Φ proof of S is a PK-proof of S from Φ and axioms of PK.

(also written as $\Phi \vdash S$)

Theorem. Let Φ be a set of (possibly infinite) sequents. Then $\Phi \models S$ iff S has a (finite) PK-Φ proof

$\Phi \models S$ iff $\Phi \vdash S$

Propositional Compactness

Theorem (Form 2, see notes for 2 other equivalent forms)

Let Φ be a set of (possibly infinite) formulas

$\Phi \models A$ iff A is a logical consequence of a finite subset of Φ

We'll assume this for now and prove it after
Proof of 3 equivalent forms of Compactness as homework

Proof (Derivational Soundness/completeness)

By compactness, it suffices to prove the case where Φ is finite

- Let $\Phi = \{S_1, .., S_k\}$, and suppose $\Gamma \Rightarrow \Delta$ is a logical consequence of $\{S_1, .., S_k\}$. Thus

  $$(*) \quad \Gamma, A_{S_1}, .., A_{S_k} \Rightarrow \Delta \quad \text{is valid}$$

- Thus by PK completeness, $(*)$ has a PK proof
- Derive $\Gamma \Rightarrow \Delta$ from $(*)$ and $\Rightarrow A_{S_1}, .., A_{S_k}$

Derive $\Gamma \to \Delta$ from $\{\to A_{s_1}, \to A_{s_2}, \to A_{s_3}, (*)\}$

PK proof of

$\Gamma, A_{s_1}, A_{s_2}, A_{s_3} \to \Delta$

$$\frac{\to A_{s_1}}{\Gamma, A_{s_2}, A_{s_3} \to \Delta, A_{s_1}}$$ (weakening)

$\Gamma, A_{s_2}, A_{s_3} \to \Delta$ (cut)

$$\frac{\to A_{s_2}}{\Gamma, A_{s_3} \to \Delta, A_{s_2}}$$ (weakening)

$$\frac{\hat{\Gamma}, A \to \hat{\Delta} \qquad \hat{\Gamma} \to A, \hat{\Delta}}{\Gamma \to \Delta}$$

$\Gamma, A_{s_3} \to \Delta$ (cut)

$$\frac{\to A_{s_3}}{\Gamma \to A_{s_3}, \Delta}$$ (weakening)

$\Gamma \to \Delta$

$\Gamma \to \Delta$ (cut)

Proof (Propositional Compactness)

Suppose $\Phi \vDash A$. Then $\overbrace{\Phi, \neg A}^{\Psi}$ is unsatisfiable

<u>show</u>: If Ψ is UNSAT, then some finite subset of Ψ is UNSAT (Form 1)

<u>Pf sketch</u> Assume the set of underlying atoms in Ψ is countable: $p_1, p_2, \ldots$

- Make decision tree T that queries p_1 at layer 1, then p_2 at layer 2, etc.

- Each path in T corresponds to a complete truth assignment

- Prune T to T':

  For every node v of T, remove subtree rooted below v if partial truth assignment from root to v falsifies some formula $f \in \varphi$. Label v by f

- Every path in T' is finite (since φ unsat, so $\forall$truth ass to all vars, some $f \in \varphi$ is falsified, and each $f \in \varphi$ is finite)

- By König's Lemma, T' is finite

König's Lemma If T' is a rooted binary tree, where every branch/path of T' is finite, Then T' is finite.

- Thus, the formulas $\Psi' \subseteq \Psi$ labelling the leaves of T' form a finite subset of Ψ, and thus Ψ' is UNSAT & finite subset of Ψ.