Announcements

Today: Finish 2^{nd} Incompleteness Thm

Review for Test 2

HW 2 solutions

Next week: Last class (Test 2)

Recap: we wanted to prove

$\exists\Delta_0$ (Exists-Delta) Theorem every r.e. relation is represented by a $\exists\Delta_0$ formula

which followed by **Main Lemma**:

f total, computable $\Rightarrow R_f$ is a $\exists\Delta_0$ relation

Recap: **First Incompleteness Theorem**

1st Incompleteness Theorem: TA is NOT axiomatizable

That is, any sound, axiomatizable theory is incomplete.

→ PA is axiomatizable. So assuming PA is sound, it is incomplete (so there are sentences A such that neither A or $\neg A$ is provable from axioms of PA.)

Tarski Theorem

Define the predicate $Truth \subseteq \mathbb{N}$

$$Truth = \{ n \mid n \text{ encodes a sentence } \langle n \rangle \in TA \}$$

Then Truth is not arithmetical.

By $\exists\Delta_0$-Theorem (every r.e. set/language is arithmetical)

this implies that Truth is Not r.e.

High Level idea of Proof:

Formulate a sentence "I am false"

which is self-contradictory

Stronger Version of $\exists\Delta_0$ Thm

<u>Recall</u>

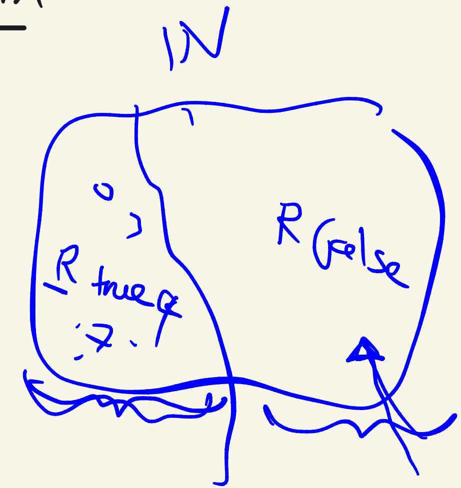
$R(x)$ is represented in TA by $A(x)$ if

$$\forall n \in \mathbb{N} \quad R(n) \Leftrightarrow TA \models A(S_n)$$

<u>Stronger version</u>:

$R(x)$ is represented in RA by $A(x)$ if

$$\forall n \in \mathbb{N} \quad R(n) \Leftrightarrow RA \models A(S_n)$$



RA Representation Theorem

Every r.e. relation is represented in RA by an $\exists\Delta_0$ formula

Corollaries of RA Representation Theorem

1. RA is Not recursive (Not decidable) Theory

   Pf sketch: K is r.e. but Not recursive
   K r.e. $\Rightarrow$ it is represented in RA by some $\exists\Delta_0$-formula A
   If RA recursive then K recursive. Contradiction

2. VALID is not recursive (Not decidable)

   Pf idea: RA is finitely axiomatizable!
   $A \in RA \Leftrightarrow P1 \wedge \dots \wedge P9 \Rightarrow A$ is valid
   so membership in RA is reducible to membership in VALID

Stronger Version of Incompleteness Thm

Recall

$R(\vec{x})$ is represented by an $\exists\Delta_0$ formula $A(\vec{x})$ if

$$\forall \vec{a} \in \mathbb{N} \quad R(\vec{a}) \Leftrightarrow TA \models A(S_{\vec{a}})$$

Stronger version:

$R(\vec{x})$ is represented in RA by $A(\vec{x})$ if

$$\forall \vec{a} \in \mathbb{N} \quad R(\vec{a}) \Leftrightarrow RA \models A(S_{\vec{a}})$$

RA Representation Theorem

Every r.e. relation is represented in RA by an $\exists\Delta_0$ formula

Incompleteness Theorems

Theorem ①

There is a specific sentence, "I am not provable" $\equiv g$ such that:

(a) PA consistent $\rightarrow$ PA $\nvdash g$

(b) PA consistent $\rightarrow$ PA $\nvdash \neg g$

This part called the 1st incompleteness theorem

Theorem ② (2nd Incompleteness theorem)

PA consistent $\Rightarrow$ PA cannot prove con(PA)

Proof of theorem ①

- $\text{Proof}(a, b) \subseteq \mathbb{N} \times \mathbb{N}$ is true iff b codes an LK-Γ_{PA} proof of the sentence coded by a
- Let $n = n(F)$ code the formula $F(x)$ with one free variable x.
  Then $\text{Proof}(d(n), m) \subseteq \mathbb{N} \times \mathbb{N}$ is true iff m codes an LK-Γ_{PA} proof of the sentence $F(s_n)$
- By RA Representation Thm, let $A(x)$ represent $\exists y \, \text{Proof}(d(x), y)$ in RA (& hence in PA)
- $\therefore \forall n \quad TA \models \exists y \, \text{Proof}(d(n), y) \iff PA \vdash A(s_n) \qquad (*)$

Proof of Theorem 1

- $\text{Proof}(a,b) \subseteq \mathbb{N}\times\mathbb{N}$ is true iff b codes an LK-Γ_{PA} proof of the sentence coded by a
- Let $n = n(F)$ code the formula $F(x)$ with one free variable x.
  Then $\text{Proof}(d(n), m) \subseteq \mathbb{N}\times\mathbb{N}$ is true iff m codes an LK-Γ_{PA} proof of the sentence $F(s_n)$
- By RA Representation Thm, let $A(x)$ represent $\exists y\, \text{Proof}(d(x), y)$ in RA (& hence in PA)
- $\therefore \forall n \quad TA \models \exists y\, \text{Proof}(d(n), y) \Leftrightarrow PA \vdash A(s_n)$ $(*)$
- Let $e = \#\neg A(x)$; $g = \neg A(s_e)$; $d(e) = \# \neg A(s_e)$

Says that "I am not provable" since $\neg A(s_e)$ says the formula encoded by $d(e)$ -- which is g -- is not provable in PA

Proof of Theorem 1

- By RA Representation Thm, let $A(x)$ represent $\exists y \, \mathrm{Proof}(d(x), y)$ in PA $(\dagger)$
- So $\forall n$: $TA \models \exists y \, \mathrm{Proof}(d(n), y) \Longleftrightarrow PA \vdash A(s_n)$ $(*)$
- Let $e = \# \neg A(x)$; $g = \neg A(s_e)$; $d(e) = \# \neg A(s_e) = \# g$

Theorem 1a $PA \vdash g \Rightarrow PA$ is not consistent

PF Suppose $PA \vdash g$
Then sentence number $d(e)$ is provable, so $\exists y \, \mathrm{Proof}(d(e), y)$ is true,
Thus $PA \vdash A(s_e)$ by left-to-rt direction of $(*)$
Thus $PA \vdash \neg g$ and $PA \vdash g$ so PA not consistent

Proof of Theorem ①

- By RA Representation Thm, let $A(x)$ represent $\exists y \, Proof(d(x), y)$ in PA (†)
- So $\forall n$: $TA \models \exists y \, Proof(d(n), y) \Longleftrightarrow PA \vdash A(s_n)$ (*)
- Let $e = \# \neg A(x)$; $g = \neg A(s_e)$; $d(e) = \# \neg A(s_e)$

1b

Theorem $PA \vdash \neg g \Rightarrow PA$ is not consistent

PF Suppose $PA \vdash \neg g$. ie. PA proves $A(s_e)$

Then $\exists y \, Proof(d(e), y)$ is true by rt-to-left direction of (*)

So PA proves $\neg A(s_e)$

So $PA \vdash g$ and $PA \vdash \neg g$, so PA **not** consistent

$\therefore$ PA consistent $\Rightarrow$ PA does not prove g + PA does not prove $\neg g$

Formulating consistency in PA

Let $B(x,y)$ be a $\exists\Delta_0$ formula that represents $\mathrm{Proof}(x,y)$ in RA (and thus also in PA)

Then for every sentence C

$$PA \vdash C \iff PA \vdash \exists y \, B(\#C, y)$$

$B(\#C, y)$ stands for $B(S_{\#C}, y)$

Define $con(PA) \stackrel{d}{=} \neg \underbrace{\exists y \, B(\#0 \neq 0, y)}$

says there is no PA proof of $0 \neq 0$ (a false statement)

Theorem ② If PA is consistent, then $PA \nvdash con(PA)$

Proof :

$\boxed{\begin{array}{l}\textbf{Main Lemma}: \; PA \vdash (con(PA) \supset g) \\ \left[\text{recall } g \overset{d}{=} \neg A(s_e),\; e = \# \neg A(x) \text{ says "I am not provable"}\right]\end{array}}$

If $PA \vdash con(PA)$ by main Lemma $PA \vdash g$

But by previous theorem
PA consistent $\Rightarrow PA \nvdash g$

$\boxed{\therefore \text{ PA consistent } \Rightarrow PA \nvdash con(PA)}$

It is left to prove:

$$\boxed{\begin{array}{l}\underline{\text{Main Lemma}}:\ PA \vdash con(PA) \supset g \\ \left[\text{recall } g \stackrel{d}{=} \neg A(S_e),\ e = \#\neg A(x) \text{ says "I am not provable"}\right]\end{array}}$$

Main step is to show that PA can prove Theorem 1 part (a)

Theorem 1a: consistency of PA $\Rightarrow$ PA cannot prove g

and therefore g is true

(since g states "I am not provable in PA")

Review for Test 2

1. completeness of LK, derivational completeness

2. computability:

   Recursive, r.e. languages/sets

   Diagonalization (to show some language not r.e.)

   Reductions (to classify other languages)

   Dovetailing

   Can assume HALT, K : r.e., not recursive

   $D = \overline{K}$: not r.e.

Properties:

L is r.e., $\overline{L}$ is r.e.

$\Rightarrow L$ is rec.

3. Incompleteness

- Defns: theory, consistent, sound, axiomatizable

- A relation $R(x)$ is represented by a ($\exists\Delta_0$) formula $A(x)$ means:

  $$\forall n \in \mathbb{N} \quad R(n) \text{ is true iff } TA \models A(S_n)$$

- A relation $R(x)$ is represented in a theory Σ by $A(x)$ means:

  $$\forall n \in \mathbb{N} \quad R(n) \text{ is true iff } \Sigma \vdash A(S_n)$$

$\exists\Delta_0$ Theorem Every r.e. relation is represented by a $\exists\Delta_0$ formula

Converse to $\exists\Delta_0$ Thm also holds: every $\exists\Delta_0$ formula $A(x)$ is r.e.

Corollaries

1. TA is not axiomatizable
(Tarski's Thm: TA not arithmetical so not r.e.)

2. Every sound, axiomatizable theory is incomplete

Important Theories: PA, RA

<u>RA Representation Theorem</u> every r.e. relation is represented in RA by a $\exists\Delta_0$ formula

<u>Strong RA Representation Thm</u> every recursive relation is <u>strongly</u> represented in RA by a $\exists\Delta_0$ formula

strongly: $R(n)$ is true $\Longleftrightarrow$ $RA \vdash A(S_n)$
$\neg R(n)$ is true $\Longleftrightarrow$ $RA \vdash \neg A(S_n)$

Corollaries (of RA representation Thm)

1. Every consistent extension of RA is undecidable

2. VALID is undecidable

Incompleteness Thms (Today)

$$\left.\begin{aligned}
&\text{1a} \quad \text{PA consistent} \Rightarrow \text{PA} \nvdash g \\
&\text{1b} \quad \text{PA consistent} \Rightarrow \text{PA} \nvdash \neg g
\end{aligned}\right\} \therefore \text{PA consistent} \Rightarrow \text{PA incomplete}$$

2\. PA consistent $\Rightarrow$ PA $\nvdash$ Con(PA)

* Show: PA $\vdash$ Con(PA) $\rightarrow g$

By *, PA $\vdash$ Con(PA) $\Rightarrow$ PA $\vdash g$, contradicting 1a

Test 2

- I'll put 2 practice tests on web.
- ~40% on computability
- ~20% short answer
- ~40% completeness, compactness, incompleteness

Thank you!

Good Luck with finals & enjoy your break!

Email me if you have any q's about this class or more generally.

Formulating consistency in PA

Let $B(x,y)$ be a $\exists\Delta_0$ formula that represents Proof(x,y) in RA (and thus also in PA)

Then for every sentence C

$$PA \vdash C \iff PA \vdash \exists y \underbrace{B(\#C, y)}_{\text{stands for } B(S_{\#C}, y)}$$

Then $PA \vdash A(S_n) \supset \exists y\, B(S_{d(n)}, y)$

[recall $A(x)$ represents $\exists y\, B(d(x), y)$]

Define $con(PA) \stackrel{d}{=} \neg \exists y\, B(\#0{\neq}0, y)$