Announcements

HW2 out last week

Recap: we wanted to prove

$\exists\Delta_0$ (Exists-Delta) Theorem every r.e. relation is represented by a $\exists\Delta_0$ formula

which followed by **Main Lemma**:

f total, computable $\Rightarrow R_f$ is a $\exists\Delta_0$ relation

Proof of Main Lemma (R_f = graph(f) is an r.e. relation)

Definition β-function

$$\beta(c, d, i) = rm(c, d(i+1)+1) \quad \text{where } rm(x, y) = x \bmod y$$

Lemma 0. $\forall n, r_0, r_1, \ldots, r_n \ \exists c, d$ such that $\forall i \le n \ \beta(c, d, i) = r_i$

the pair (c, d) represents the sequence $\underbrace{r_0\, r_1, \ldots r_n}_{\text{entire tableaux of TM configuration}}$ using β

Lemma 1 graph(β) is a Δ_0 relation

Recap: **First Incompleteness Theorem**

$\boxed{\text{1st Incompleteness Theorem:}}$ TA is Not axiomatizable

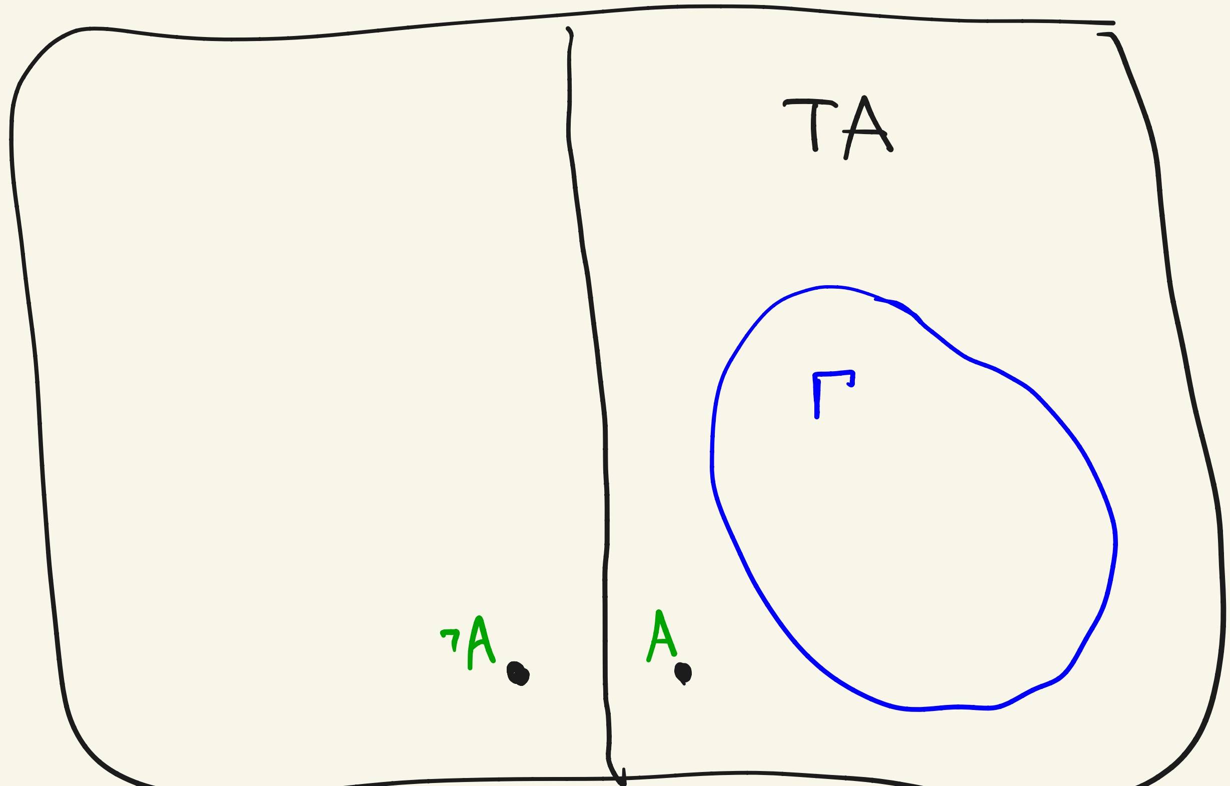
That is, any sound, axiomatizable theory is incomplete.

→ PA is axiomatizable. So assuming PA as sound, it is incomplete (so there are sentences A such that neither A or $\neg A$ is provable from axioms of PA.)

$$K^c = \{\underbrace{\ulcorner x \urcorner \mid \{x\}(x) \text{ doesn't halt}}_{F(\tilde{x})}\} \qquad K$$

Φ_0:

all $\mathcal{L}_A$ sentences



Γ sound and axiomatizable $\Longrightarrow \exists A, \neg A \notin \Gamma$

Tarski Theorem

Define the predicate $Truth \subseteq \mathbb{N}$

$$Truth = \{ n \mid n \text{ encodes a sentence } \langle n \rangle \in TA \}$$

Then Truth is not arithmetical.

By $\exists\Delta_0$-Theorem (every r.e. set/language is arithmetical)

this implies that Truth is not r.e.

High Level idea of Proof:

Formulate a sentence "I am false"

which is self-contradictory

Pf of Tarski's Thm

Let $sub(m,n) = \begin{cases} 0 \text{ if } m \text{ is not a legal encoding of a formula} \\ \text{otherwise let } m \text{ encode the formula } A(x) \text{ with free variable } x. \text{ Then } sub(m,n) = m' \text{ where } m' \text{ encodes } A(S_n) \end{cases}$

[sub(m,n): decode m, plug in n + re-encode]

Let $d(m) = sub(m,m)$

$\begin{cases} d(m) = 0 \text{ if } m \text{ not a legal encoding.} \\ \text{ow say } m \text{ encodes } A(x). \\ \text{then } d(m) = m' \text{ where } m' \text{ encodes } A(S_m) \end{cases}$

clearly sub, d are both computable
so by $\exists \Delta_0$-theorem graph(sub), graph(d) are arithmetical

Proof of Tarski's Thm

Suppose that Truth is arithmetical.

Then define $R(m) = \neg \mathrm{Truth}(d(m))$

Since d, Truth both arithmetical, so is R

Let $\widetilde{R(m)}$ represent $R(m)$, and let e be the encoding of $\widetilde{R(m)}$

Then $d(e) =$ encoding of $\underbrace{\widetilde{R(s_e)}}$

encodes "I am false"

Then

$$\widetilde{R(s_e)} \in TA \iff \neg \mathrm{Truth}(d(e)) \quad \text{since } \tilde{R} \text{ represents } R$$

$$\iff \neg \widetilde{R(s_e)} \in TA \quad \text{by defn of truth}$$

($\mathrm{Truth}(d(e))$ says $R(s_e) \in TA$)

this is a contradiction since A and $\neg A$ cannot both be in TA #

$R(x)$ represented by a formula $A(x)$ if:

$$\forall n \in \mathbb{N} \quad R(n) \iff TA \models A(S_n)$$

$$S_n = \underbrace{SS \ldots S}_{n} 0$$

term corresponding to the number n

PEANO ARITHMETIC

P1. $\forall x\ (sx \neq 0)$

P2. $\forall x \forall y\ (sx = sy \supset x = y)$ — s is 1-1

P3. $\forall x\ (x + 0 = x)$

P4. $\forall x \forall y\ (x + sy = s(x+y))$

$\Big\}$ define $+$

P5. $\forall x\ (x \cdot 0 = 0)$

P6. $\forall x \forall y\ (x \cdot sy = (x \cdot y) + x)$

$\Big\}$ define $\cdot$

$$\mathrm{IND}(A(x)) : \forall y_1 \ldots \forall y_k \left[\left(A(0) \wedge \forall x\ (A(x) \supset A(sx)) \right) \supset \forall x\ A(x) \right]$$

INDUCTION AXIOMS: All sentences $\mathrm{IND}(A(x))$ for all formulas A whose free variables are $y_1 \ldots y_k, x$

$$\Gamma_{PA} = \{P_1, \ldots, P_6\} \cup \{\text{INDUCTION AXIOMS}\}$$

1. Γ_{PA} is recursive

2. PA is sound + axiomatizable (so incomplete)

3. PA still strong enough to prove all of standard number theory

Robinson's Arithmetic RA

Axioms $\{P1, \dots, P6\}$ of PA plus P7, P8, P9

P7 : $(\forall x \; x \leq 0 \supset x = 0)$

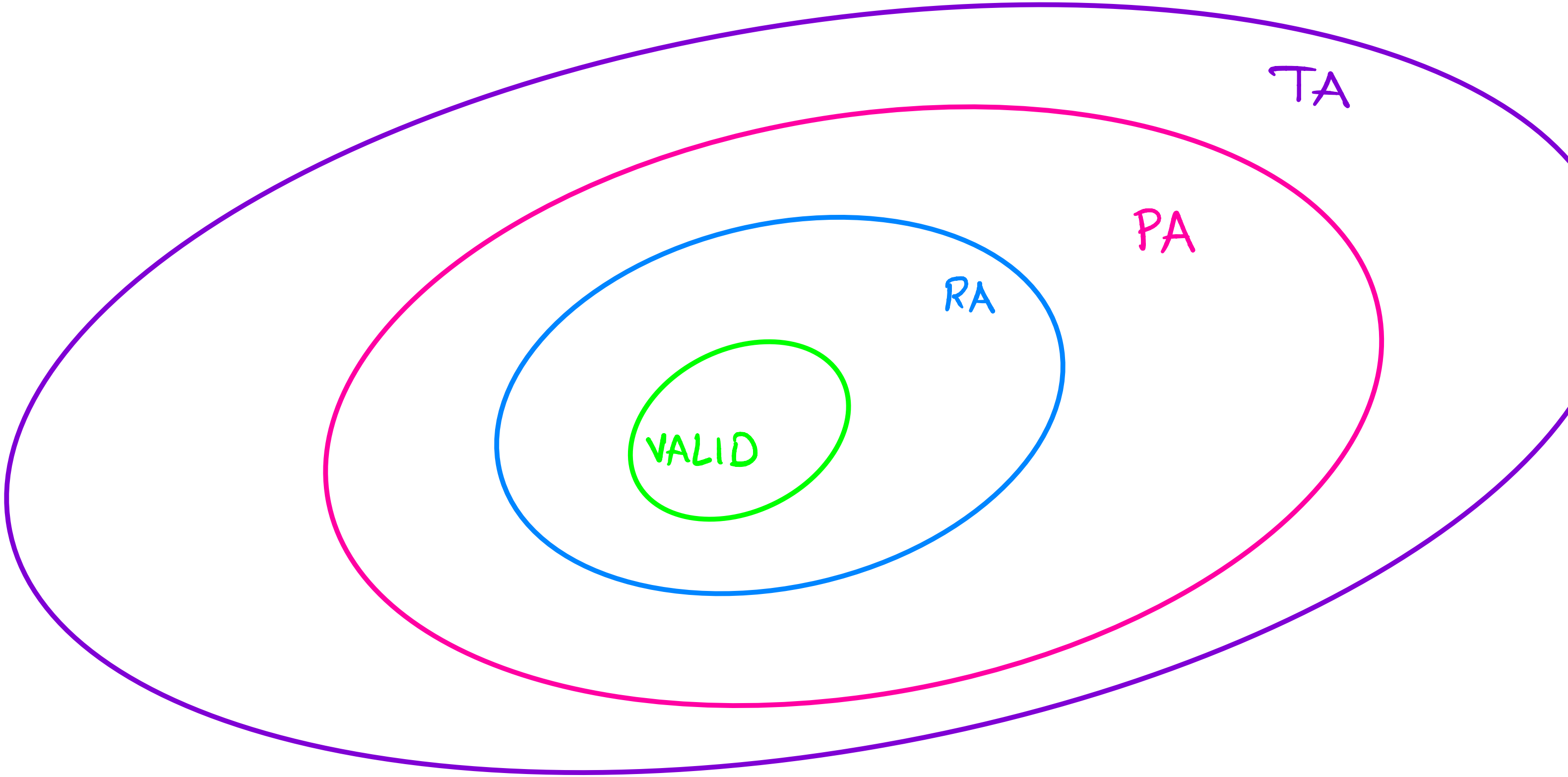
P8 : $\forall x \forall y \, (x \leq sy \supset (x \leq y \vee x = sy))$

P9 : $\forall x \forall y \; (x \leq y \vee y \leq x)$

where $t_1 \leq t_2$ abbreviates $\exists z \, (t_1 + z = t_2)$

FACTS
1. $RA \subseteq PA$
2. RA finitely axiomatizable
3. Over extended language $\mathcal{L}_{A,\leq}$, RA axioms are $\forall$sentences

SOUND, CONSISTENT THEORIES OF ARITHMETIC



Stronger Version of $\exists\Delta_0$ Thm

Recall

$R(x)$ is represented in TA by $A(x)$ if

$\forall n \in \mathbb{N} \quad R(n) \Leftrightarrow TA \models A(S_n)$

Stronger version:

$R(x)$ is represented in RA by $A(x)$ if

$\forall n \in \mathbb{N} \quad R(n) \Leftrightarrow RA \models A(S_n)$

RA Representation Theorem

Every r.e. relation is represented in RA by an $\exists\Delta_0$ formula

Corollaries of RA Representation Theorem

1. RA is Not recursive (Not decidable)
   Theory

   Pf sketch: K is r.e. but Not recursive

   K r.e. $\Rightarrow$ it is represented in RA by some $\exists\Delta_0$-formula A

   If RA recursive then K recursive. Contradiction

2. VALID is not recursive (Not decidable)

   Pf idea: RA is finitely axiomatizable!

   $A \in RA \iff P1 \wedge \dots \wedge P9 \Rightarrow A$ is valid

   so membership in RA is reducible to membership in VALID

"RA" is recursive

the set of sentences of RA is decidable

i.e. $\exists$ TM M always halts and s.t.

$\forall$ sentence A, $M(\tilde{A})$ accepts iff $A \in RA$

encoding of A

$K \stackrel{\mathrm{def}}{=} \{ x \mid \langle x \rangle (x) \text{ halts} \}$

TM x on input x

$RA \vDash F(S_x) \iff K(x) = 1$

Let F represent K:

F has one free variable x

RA Representation Theorem

Every r.e. relation is represented in RA by an $\exists\Delta_0$ formula

Proof idea

$\boxed{\text{Main Lemma}}$: every Δ_0-sentence in TA is provable in RA

Assuming Main Lemma, Let $R(\vec{x})$ be an r.e. relation.

By Exists-Delta Theorem, $R(\vec{x})$ is represented (in TA) by some $\exists\Delta_0$-formula $A(\vec{x})$

$$\text{so } \forall \vec{a} \in \mathbb{N}^k \quad R(\vec{a}) \iff \exists y\, \underbrace{A(S_{\vec{a}}, y)}_{\Delta_0} \in TA$$

$\therefore$ By soundness of RA & since every Δ_0 sentence of TA is provable in RA

$$R(\vec{a}) \iff RA \vdash \exists y\, A(S_{\vec{a}}, y) \qquad \exists y\; RA \vdash A(S_{\vec{a}}, S_y)$$

So $\exists y\, A(\vec{x}, y)$ represents $R(\vec{x})$ in RA

PROOF (SKETCH) OF Main Lemma

Main Lemma: every Δ_0-sentence in TA is provable in RA

Proof By induction on number of logical symbols in A

For convenience easier to work in $RA_{\leq}$: $\leq$ new relation symbol,

Axioms P1-P9 plus new axiom P0

$$P0: \quad \forall x \forall y \, (x \leq y \leftrightarrow \exists z \, (x + z = y))$$

Δ_0 sentence:

$$\exists x \leq \underbrace{\left[(SS0 + SSS0) \cdot SSSSS0\right]}_{25} \forall y \leq (x + x) SSS0 \;\; A(x, y)$$

PROOF (SKETCH) OF Main Lemma

$\boxed{\text{Main Lemma}}$: every Δ_0-sentence in TA is provable in RA

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Base Case A: $t = u$, $\neg(t = u)$, $t \leq u$, $\neg(t \leq u)$

Lemma A1: $RA_{\leq} \vdash S_m + S_n = S_{m+n}$
$\vdash S_m \cdot S_n = S_{m \cdot n}$

Lemma A: t closed (No variables) and $TA \vdash t = S_n$, then $RA_{\leq} \vdash t = S_n$

Lemma B: $\forall m \neq n \quad RA_{\leq} \vdash S_n \neq S_m$

Lemma C: $RA_{\leq} \vdash \forall x \, (x \leq S_n \supset (x = 0 \vee x = S_1 \vee x = S_2 \vee \ldots \vee x = S_n))$

ex.

$$t: SS0 + \underbrace{SS\cdots}_{n-2}0$$

$$TA \models t = S_n$$

$$\underbrace{SSS\cdots S}_{n}0$$

PROOF (SKETCH) OF Main Lemma

$\boxed{\text{Main Lemma}}$: every Δ_0-sentence in TA is provable in RA

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Induction step

① $A = A_1 \vee A_2$, $A_1 \wedge A_2$ Apply ind. hyp.

② $A = \forall x \leq t \, B(x)$

- t closed so by Lemma A $RA_{\leq} \vdash t = s_n$ for some fixed n
- Show $RA_{\leq} \vdash \forall x \leq n \, B(x)$

By Lemma C $RA_{\leq} \vdash x \leq n \supset (x = 0 \vee x = s_1 \vee \ldots \vee x = s_n)$

By induction $RA_{\leq} \vdash B(c)$ for all $c \leq n$

Put together to derive $RA_{\leq} \vdash \forall x \leq n \, B(x)$

PROOF (SKETCH) OF Main Lemma

Main Lemma: every Δ_0-sentence in TA is provable in RA

Proof By induction on number of logical symbols in A

For convenience easier to work in $RA_{\leq}$: $\leq$ new relation symbol,

Axioms P1-P9 plus new axiom P0

P0: $\forall x \forall y \, (x \leq y \leftrightarrow \exists z \, (x + z = y))$

Induction step

(1) $A = A_1 \vee A_2$, $A_1 \wedge A_2$ Apply ind. hyp.

(2) $A = \forall x \leq t \; B(x)$

- t closed so by Lemma A $RA_{\leq} \vdash t = S_n$ for some fixed n
- Show $RA_{\leq} \vdash \forall x \leq n \; B(x)$

  By Lemma C $RA_{\leq} \vdash x \leq n \supset (x = 0 \vee x = S_1 \vee \ldots \vee x = S_n)$

  By induction $RA_{\leq} \vdash B(c)$ for all $c \leq n$

  Put together to derive $RA_{\leq} \vdash \forall x \leq n \; B(x)$

(3) $A = \exists x \leq t \; B(x)$ proof similar to (2)

Consequences of MAIN LEMMA

① every $\exists\Delta_0$ sentence of TA is provable in RA

② The $\exists\Delta_0$ sentences of TA are r.e. but not recursive

(and the Δ_0 sentences of TA <u>are</u> recursive/decidable)

Stronger Version of Incompleteness Thm

Recall

$R(\vec{x})$ is represented by an $\exists\Delta_0$ formula $A(\vec{x})$ if

$$\forall \vec{a} \in \mathbb{N} \quad R(\vec{a}) \iff TA \models A(S_{\vec{a}})$$

Stronger version:

$R(\vec{x})$ is represented in RA by $A(\vec{x})$ if

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RA Representation Theorem

Every r.e. relation is represented in RA by an $\exists\Delta_0$ formula

EVEN Stronger Version of Incompleteness Thm

Recall

(1) $R(\vec{x})$ is represented by an $\exists\Delta_0$ formula $A(\vec{x})$ if

$$\forall \vec{a} \in \mathbb{N} \quad R(\vec{a}) \Leftrightarrow TA \models A(S_{\vec{a}})$$

(2) Stronger version:

$R(\vec{x})$ is represented in RA by $A(\vec{x})$ if

$$\forall \vec{a} \in \mathbb{N} \quad R(\vec{a}) \Leftrightarrow RA \models A(S_{\vec{a}})$$

(3) Even stronger

$R(\vec{x})$ is strongly represented in RA by $A(\vec{x})$ if

$$\forall \vec{a} \in \mathbb{N} \quad R(\vec{a}) \Rightarrow RA \models A(S_{\vec{a}})$$

$$\neg R(\vec{a}) \Rightarrow RA \models \neg A(S_{\vec{a}})$$

Strong RA Representation Theorem

Every ~~r.e.~~ recursive relation is strongly represented in RA by an $\exists\Delta_0$ formula