Completeness of System LK for Predicate Calculus

In general in this section of the notes we assume that every formula A satisfies the restriction described on page 27: All free variables of A are from the free variable list $a, b, c, ...$ and all bound variables of A are from the bound variable list $x, y, z, ...$

On page 29 we proved the soundness of LK: Every sequent provable in LK is valid. Now we want to prove completeness, which is the converse: Every valid sequent is provable in LK. This completeness theorem, originally proved by Gödel in 1930 (for a different proof system), ranks as one of the great results in logic of this century. The definition of a valid formula (page 23) is a formula that is true under all interpretations (structures), including those with an infinite universe (even an uncountably infinite universe). The completeness theorem states that for every valid formula A, this universal truth can be established by exhibiting a finite syntactic object. In our case, that object is an LK proof of the sequent $\rightarrow A$.

Actually we will prove a stronger result, a kind of derivational completeness, in order to get the first-order compactness theorem as a corollary. As in the propositional system PK, we want to generalize LK proofs to allow nonlogical axioms. We will restrict attention to the simple case in which every nonlogical axiom has the form $\rightarrow A$, where A is a formula.

Definition: If Φ is a set of formulas, then an $LK - \Phi$ proof is an LK proof in which sequents at the leaves may be either logical axioms of the form $A \rightarrow A$ (for any formula A), or nonlogical axioms of the form $\rightarrow A$, where A is in Φ.

We would like to be able to say that a sequent $\Gamma \rightarrow \Delta$ is a logical consequence of a set Φ of formulas iff there is an $LK - \Phi$ proof of $\Gamma \rightarrow \Delta$. Unfortunately the soundness direction of the assertion is false. For example, using the $\forall$ **right** rule we can derive $\rightarrow \forall x Px$ from $\rightarrow Pb$, but $\rightarrow \forall x Px$ is not a logical consequence of Pb.

We could correct the soundness statement by asserting it true for sentences, but we want to generalize this a little by introducing the notion of the universal closure of a formula or sequent.

Recall the definition (page 28) of the universal closure $\forall A$ of a formula A. Also if Φ is a set of formulas, then $\forall \Phi$ is the set of all sentences $\forall A$, for A in Φ. Notice that if A is a sentence (i.e. no free variables), then $\forall A$ is the same as A.

Exercise 1 *Prove from the definition of logical consequence (page 23) that every formula A is a logical consequence of its universal closure $\forall A$. (I.e. show that $\forall A \models A$.) Give an example of a formula A such that $A \not\models \forall x A$. Show however that for every formula A we have $\models A$ iff $\models \forall A$.*

Exercise 2 *Show that if* $\Phi \models A$ *then* $\forall\Phi \models \forall A$*. Give an example to show that the converse may be false: We could have* $\forall\Phi \models \forall A$ *but not* $\Phi \models A$*.*

Initially we study the case in which the underlying vocabulary does not contain $=$. To handle the case in which $=$ occurs we must introduce equality axioms. This will be done later.

Theorem: (Derivational Soundness and Completeness of LK) Assume that the underlying vocabulary does not contain $=$. Let Φ be a set of sequents and let $\Gamma \to \Delta$ be a sequent. Then there is an $LK - \Phi$ proof of $\Gamma \to \Delta$ iff $\Gamma \to \Delta$ is a logical consequence of $\forall\Phi$. The soundness (only if) direction holds also when the underlying vocabulary contains $=$.

Proof of Soundness: Let π be a $LK - \Phi$ proof of $\Gamma \to \Delta$. We must show that $\Gamma \to \Delta$ is a logical consequence of $\forall\Phi$. We want to prove this by induction on the number of sequents in the proof π, but in fact we need a stronger induction hypothesis, to the effect that the "closure" of $\Gamma \to \Delta$ is a logical consequence of $\forall\Phi$. So we first have to define the closure of a sequent.

Thus we define the closure $\forall S$ of a sequent S to be the closure of its associated formula (page 28): $\forall S =_{syn} \forall A_S$, where $A_S =_{syn} \bigwedge \Gamma \supset \bigvee \Delta$, when $S =_{syn} \Gamma \to \Delta$. Note that $\forall S$ is *not* equivalent to $\forall\Gamma \to \forall\Delta$ in general.

We now prove by induction on the number of sequents in π, that if π is an $LK - \Phi$ proof of a sequent S, then $\forall S$ is a logical consequence of $\forall\Phi$. It will follow by Exercise 1 above that S itself is a logical consequence of $\forall\Phi$, and so Soundness follows.

For the base case, the sequent S is either a logical axiom $A \to A$, which is valid and hence a consequence of $\forall\Phi$, or it is a nonlogical axiom $\to A$, where A is a formula in Φ. In the latter case, $\forall S$ is equivalent to $\forall A$, which of course is a logical consequence of $\forall\Phi$.

For the induction step, it is sufficient to check that for each rule of LK, the closure of the bottom sequent is a logical consequence of the closure(s) of the sequent(s) on top. With two exceptions, this statement is true when the word "closure" is omitted, and adding back the word "closure" then follows from Exercise 2 above. The two exceptions are the rules, $\forall$-**right** and $\exists$-**left**. For these, the bottom is *not* a logical consequence of the top in general, but it is not hard to show that the closures of the top and bottom are equivalent. Note however that the **Restriction** on the free variable b in these rules (page 28) is essential in order to show that the closure of the bottom is a logical consequence of the closure of the top. □

The proof of completeness is more difficult and more interesting than the proof of soundness. The basic idea is similar to the proof for the propositional system PK, but unfortunately the Inversion Principle fails for two of the new LK rules: $\exists$-**right** and $\forall$-**left**. To get around this we will have to apply these rules repeatedly using different terms t, and the contraction rule becomes necessary (at least for cut-free proofs).

The following lemma lies at the heart of this proof.

Completeness Lemma: Assume that the underlying vocabulary does not contain $=$. If

$\Gamma \to \Delta$ is a sequent and Φ is a set of formulas such that $\Gamma \to \Delta$ is a logical consequence of Φ, then there is a finite subset $\{C_1, ..., C_n\}$ of Φ such that the sequent

$$C_1, ..., C_n, \Gamma \to \Delta$$

has an LK proof π which does not use the cut rule.

Proof of Derivational Completeness from the Completeness Lemma: Let Φ be a set of formulas such that $\Gamma \to \Delta$ is a logical consequence of $\forall\Phi$. By the completeness lemma, there is a finite subset $\{C_1, ..., C_n\}$ of Φ such that

$$\forall C_1, ..., \forall C_n, \Gamma \to \Delta$$

has a cut-free LK proof π. Note that for each $i, 1 \leq i \leq n$, the sequent $\to \forall C_i$ has an $LK - \Phi$ proof from the nonlogical axiom $\to C_i$ by repeated use of the rule $\forall$-**right**. Now the proof π can be extended, using these proofs of the sequents

$$\to \forall C_1 \quad ... \to \forall C_n$$

and repeated use of the cut rule, to form an $LK - \Phi$ proof $\Gamma \to \Delta$. □

Proof of the Completeness Lemma: We loosely follow the proof of the Cut-free Completeness Theorem, pp 33-36 of [Buss]. (Warning: our definition of logical consequence differs from Buss's when the formulas in the hypotheses have free variables.) We will only prove it for the case in which the underlying first-order vocabulary $\mathcal{L}$ has a countable set (including the case of a finite set) of function and predicate symbols; i.e. the function symbols form a list $f_1, f_2, ...$ and the predicate symbols form a list $P_1, P_2, ...$ This may not seem like much of a restriction, but for example in developing the model theory of the real numbers, it is sometimes useful to introduce a distinct constant symbol e_c for every real number c; and there are uncountably many real numbers. The completeness theorem and lemma hold for the uncountable case, but we shall not prove them for this case.

For the countable case, we may assign a distinct binary string to each function symbol, predicate symbol, variable, etc. and hence assign a unique binary string to each formula and term. This allows us to enumerate all the $\mathcal{L}$-formulas in a list $A_1, A_2, ...$ and enumerate all the $\mathcal{L}$-terms in a list $t_1, t_2, ...$ Here we assume that all free variables in each formula A_i or term t_j come from the list $a, b, c, ...$ of free variables. Further we may assume that every formula occurs infinitely often in the list of formulas, and every term occurs infinitely often in the list of terms. (For example, we can take the original sequence of terms and form a new sequence $t_1, t_1, t_2, t_1, t_2, t_3, ...$ which satisfies this condition.) Finally we may enumerate all pairs $\langle A_i, t_j \rangle$, using any method of enumerating all pairs of natural numbers.

We are trying to find an LK proof of some sequent of the form $C_1, ..., C_n, \Gamma \to \Delta$, for some n. Starting with $\Gamma \to \Delta$ at the bottom, we work upwards by applying the rules in reverse, much as in the proof of the propositional completeness theorem for PK. However now we will add formulas C_i to the antecedent from time to time. Also unlike the PK case we have no inversion principle to work with (specifically for the rules $\forall$-**left** and $\exists$-**right**). Thus it may happen that our proof-building procedure may not terminate. In this case we will show how to define a structure which shows that $\Gamma \to \Delta$ is not a logical consequence of Φ.

We construct our cut-free proof tree π in stages. Initially π consists of just the sequent $\Gamma \to \Delta$. At each stage we modify π by possibly adding a formula from Φ to the antecedent of every sequent in π, and by adding subtrees to some of the leaves.

Definition: A sequent in π is said to be *active* provided it is at a leaf, and no formula occurs in both its antecedent and succedent.

Note that if some formula does occur on both sides, the sequent can be derived from a logical axiom with weakenings and exchanges.

Each stage uses one pair in our enumeration of all pairs $\langle A_i, t_j \rangle$. Here is the procedure for the next stage, in general.

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Let $\langle A_i, t_j \rangle$ be the next pair in the enumeration. We call A_i the *active* formula for this stage.

Step (1): If A_i is in Φ, then replace every sequent $\Gamma' \to \Delta'$ in π with the sequent $\Gamma', A_i \to \Delta'$.

Step (2): If A_i is atomic, do nothing and proceed to the next stage. Otherwise, modify π at the active sequents which contain A_i by applying the appropriate introduction rule in reverse, much as in the propositional completeness proof on page 13. For example, if A_i is of the form $B \vee C$, then every active sequent in π of the form $\Gamma', B \vee C, \Gamma'' \to \Delta'$ is replaced by the derivation

$$\frac{\Gamma', B, \Gamma'' \to \Delta' \qquad \Gamma', C, \Gamma'' \to \Delta'}{\Gamma', B \vee C, \Gamma'' \to \Delta'}$$

Here the double line represents a derivation involving the rule $\vee$-**left**, together with exchanges to move the principle formulas to the left end of the antecedent and back. The treatment is similar for the other propositional cases.

If A_i is of the form $\exists x B(x)$, then every active sequent of π of the form $\Gamma', \exists x B(x), \Gamma'' \to \Delta'$ is replaced by the derivation

$$\frac{\Gamma', B(c), \Gamma'' \to \Delta'}{\Gamma', \exists x B(x), \Gamma'' \to \Delta'}$$

where c is a new free variable, not used in π yet. (Also c may not occur in any formula in Φ, because otherwise at a later stage, Step (1) of the procedure might cause the variable restriction in the $\exists$-**left** rule to be violated.)

In addition, any active sequent of the form $\Gamma' \to \Delta', \exists x B(x), \Delta''$ is replaced by the derivation

$$\frac{\Gamma' \to \Delta', \exists x B(x), B(t_j), \Delta''}{\Gamma' \to \Delta', \exists x B(x), \Delta''}$$

Here the term t_j is the second component in the current pair $\langle A_i, t_j \rangle$. The derivation uses the rule $\exists$-**right** to introduce a new copy of $\exists x B(x)$, and then the rule contraction-**right** to combine the two copies of $\exists x B(x)$.

NOTE: This and the dual $\forall$-**left** case are the only two cases that use the term t_j, and the only cases that use the **contraction rules**.

The case where A_i begins with a universal quantifier is dual to the above existential case.

Step (3) If there are no active sequents remaining in π, then exit from the algorithm. Otherwise continue to the next stage.

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Exercise 3 *Carry out the case above in which A_i begins with a universal quantifier.*

If the algorithm constructing π ever halts, then π gives a cut-free proof of $\Gamma, C_1, ..., C_n \rightarrow \Delta$ for some formulas $C_1, ..., C_n$ in Φ. This is because the nonactive leaf sequents all contain a formula A which occurs on both the left and right, and each such sequent can be derived from the logical axiom $A \rightarrow A$, using weakenings and exchanges. Thus π can be extended, using exchanges, to a cut-free proof of $C_1, ..., C_n, \Gamma \rightarrow \Delta$, as desired.

It remains to show that if the above algorithm constructing π never halts, then the sequent $\Gamma \rightarrow \Delta$ is not a logical consequence of Φ. So suppose the algorithm never halts, and let π be the result of running the algorithm forever. In general, π will be an infinite tree, although in special cases π is a finite tree. In general the objects at the leaves of the tree will not be finite sequents, but because of Step (1) of the algorithm above, they will be of the form $\Gamma', C_1, C_2, ... \rightarrow \Delta'$, where $C_1, C_2, ...$ is an infinite sequence of formulas containing all formulas in Φ, each repeated infinitely often (unless Φ is empty). We shall refer to these infinite pseudo-sequents as just "sequents".

If π has only finitely many nodes, then at least one leaf node must be active (and contain only atomic formulas), since otherwise the algorithm would terminate. In this case, let β be a path in π from the root extending up to this active node. If on the other hand π has infinitely many nodes, then by König's Lemma (page 16), there must be an infinite branch β in π starting at the root and extending up through the tree. Thus in either case, β is a branch in π starting at the root, extending up through the tree, and such that all sequents on β were once active, and hence have no formula occurring on both the left and right.

Term model construction

We use this branch β to construct a structure $\mathcal{M}$ and an object assignment σ which satisfy every formula in Φ, but falsify the sequent $\Gamma \rightarrow \Delta$ (so $\Gamma \rightarrow \Delta$ is not a logical consequence of Φ).

The universe M of $\mathcal{M}$ is the set of all $\mathcal{L}$-terms t (which contain only "free" variables $a, b, c, ...$). (Hence $\mathcal{M}$ is called a *term model*.)

The object assignment σ just maps every variable a to itself.

The interpretation $f^{\mathcal{M}}$ of each k-ary function symbol f is defined so that $f^{\mathcal{M}}(r_1, ..., r_k)$ is the term $fr_1...r_k$, where $r_1, ..., r_k$ are any terms (i.e. any members of the universe).

The interpretation $P^{\mathcal{M}}$ of each k-ary predicate symbol P is defined by letting $P^{\mathcal{M}}(r_1, ..., r_k)$ hold iff the atomic formula $Pr_1...r_k$ occurs in the antecedent (left side) of some sequent in the branch β.

Exercise 4 *Prove by structural induction that for every term t, $t^{\mathcal{M}}[\sigma] = t$.*

Claim: For every formula A, if A occurs in some antecedent in the branch β, then $\mathcal{M}$ and σ satisfy A, and if A occurs in some succedent in β, then $\mathcal{M}$ and σ falsify A.

Since the root of π is the sequent $\Gamma, C_1, C_2, ... \to \Delta$, where $C_1, C_2, ...$ contains all formulas in Φ, it follows that $\mathcal{M}$ and σ satisfy Φ and falsify $\Gamma \to \Delta$.

We prove the Claim by structural induction on formulas A. For the base case, if A is an atomic formula, then by the definition of $P^{\mathcal{M}}$ above, A is satisfied iff A occurs in some antecedent of β. But no atomic formula can occur both in an antecedent of some node in β and in a succedent (of possibly some other node) in β, since then these formulas would persist upward in β so that some particular sequent in β would have A occurring both on the left and on the right. Thus if A occurs in some succedent of β, it is not satisfied by $\mathcal{M}$ and σ.

For the induction step, there is a different case for each of the ways of constructing a formula from simpler formulas (see the definition of formula, page 19). In general, if A occurs in some sequent in β, then A persists upward in every higher sequent of β until it becomes the active formula ($A =_{syn} A_i$). Each case is handled by the corresponding introduction rule used in the algorithm. For example, if A is of the form $B \vee C$ and A occurs on the left of a sequent in β, then the rule $\vee$-**left** is applied in reverse (see page 34), so that when β is extended upward either it will have some antecedent containing B or one containing C. In the case of B, we know that $\mathcal{M}$ and σ satisfy B by the induction hypothesis, and hence they satisfy $B \vee C$. (Similarly for C.)

Now consider the interesting case in which A is $\exists x B(x)$ and A occurs in some succedent of β. See page 34 to find out what happens when A becomes active. The path β will hit a sequent with $B(t_j)$ in the succedent, and by the induction hypothesis, $\mathcal{M}$ and σ falsify $B(t_j)$. But this succedent still has a copy of $\exists x B(x)$, and in fact this copy will be in *every* succedent of β above this point. Hence *every* $\mathcal{L}$-term t will eventually be of the form t_j and so the formula $B(t)$ will occur as a succedent on β. (This is why we assumed that every term appears infinitely often in the sequence $t_1, t_2, ...$) Therefore $\mathcal{M}$ and σ falsify $B(t)$ for every term t (i.e. for every element in the universe of $\mathcal{M}$). Therefore they falsify $\exists x B(x)$, as required.

This and the dual case in which A is $\forall x B(x)$ and occurs in some antecedent of β are the only subtle cases. All other cases are straightforward. $\Box$.

Exercise 5 *Consider the sequent*

$$Pe, \forall x(\neg Px \vee Pffx) \rightarrow Pfe$$

where e is a constant and f is a unary function symbol. This sequent is not valid, and hence the procedure described in the proof of the Completeness Lemma will not terminate with an LK proof. In this case it is sufficient to consider the set

$$T = \{e, fe, ffe, fffe, \ldots\}$$

of terms. Carry out the construction (repeatedly apply Step (2)) in the proof to get an LK tree π *with an infinite branch, where the terms used in the* $\forall$-**Left** *rule are restricted to the set* T*. (You do not need to describe all of* π*, but describe one infinite path in* π *and describe the resulting term model. Show that it falsifies the sequent.)*

Exercise 6 *What goes wrong in the above proof if* $\Gamma \rightarrow \Delta$ *contains =?*

Exercise 7 *Show how to use a contraction rule to get a cut-free proof of the sequent*

$$\forall x(\neg Px \vee Pfx) \rightarrow \forall x(\neg Px \vee Pffx)$$

We have already proved derivational completeness from the Completeness Lemma (page 33), but now we mention a stronger form of derivational completeness (*anchored completeness*) which shows that the cut formulas in the LK derivation can be restricted to formulas in the hypothesis set Φ, provided that Φ is closed under substitution of terms for variables (i.e., if $A(b)$ is in Φ, and t is a proper term (page 27) then $A(t)$ is also in Φ.) Note that if Φ is a set of sentences, it is automatically closed under substitution of terms for variables.

Anchored Completeness Theorem: Assume that the underlying vocabulary is countable and does not contain =. Suppose that Φ is a set of formulas closed under substitution of terms for variables. Suppose that $\Gamma \rightarrow \Delta$ is a sequent that is a logical consequence of $\forall\Phi$. Then there is an $LK - \Phi$ proof of $\Gamma \rightarrow \Delta$ in which the cut rule is restricted so that the only cut formulas are formulas in Φ.

Note that if all formulas in Φ are sentences, then the above theorem follows easily from the Completeness Lemma, since in this case $\forall\Phi$ is the same as Φ. However if formulas in Φ have free variables, then apparently the cut rule must be applied to the closures $\forall C$ of formulas in Φ (as opposed to C itself) in order to get an $LK - \Phi$ proof of $\Gamma \rightarrow \Delta$, as explained on page 33. To see that the cut formulas can be taken to be in Φ instead of closures of formulas in Φ, the proof of the Completeness Lemma can be modified, as explained in the following exercise.

Exercise 8 *Show how to prove the above theorem by modifying the proof of the Completeness Lemma according to a),...,d) below.*

a) the definition of *active sequent* on page 34 must be modified, since now we're allowing nonlogical axioms in π. Give the precise new definition.

b) Step (1) of the procedure on page 34 must be modified, because now we're looking for a derivation of $\Gamma \rightarrow \Delta$ from nonlogical axioms, rather than a proof of $C_1, ..., C_n, \Gamma \rightarrow \Delta$. Describe the modification. (We still need to bring formulas A_i of Φ somehow into the proof, and your modification will involve adding a short derivation to π.)

c) the restriction given on page 34 for the case $\exists x B(x)$ on the left, that the variable c must not occur in any formula in Φ, must be dropped. Explain why.

d) Explain why the structure $\mathcal{M}$ and object assignment σ, described on page 35, satisfy $\forall\Phi$. This should follow from the Claim on page 36, and your modification of Step (1), which should ensure that each formula in Φ occurs in the antecedent of some sequent in every branch in π. Conclude that $\Gamma \rightarrow \Delta$ is not a logical consequence of $\forall\Phi$ (when the procedure does not terminate).