Announcements

NEW DATES

- Problem Set 2 : Due MON NOV 15
- Test 2 : MON NOV 22
- Problem Set 3 : Due MON Dec 13

TODAY:

- Corollaries of completeness
- Dealing with Equality
- Theories of Arithmetic

Corollaries of Completeness

(1) **Lowenheim-Skolem Theorem**. Let $\mathcal{L}$ be countable, Φ a set of sentences over $\mathcal{L}$.

Φ satisfiable $\Rightarrow$ Φ is satisfiable in a countable universe.

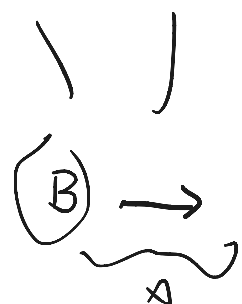
Proof Follows from completeness proof. Let Φ be satisfiable

Let $A = \;\longrightarrow$ (empty sequent is unsatisfiable)

Then $\Phi \not\models A$, so proof of completeness constructs a countable model where Φ is satisfiable. $\boxtimes$

ϕ finite

$B \in \Phi$



Algorithm for finding a Φ-LK proof of $\to A$

Enumerate all $\langle B, t\rangle$ $\quad$ B a formula over $\mathcal{L}$, t is a term

s.t. every B, t occur infinitely often

Step i use $\langle B,t\rangle_i$:

1. If $B \in \Phi$ then add B to LHS of all sequents

2. If B not atomic, then:

   For every active sequent containing B, apply the appropriate rule in reverse.

   Use t if $B = \exists x A(x)$ & occurs on Rt

   Use t if $B = \forall x A(x)$ & occurs on left

Corollaries of Completeness

(2) First Order Compactness Theorem.

An infinite set of first order sentences Φ is unsatisfiable if and only if some finite subset of Φ is unsatisfiable

Proof Let A be the empty sequent (or any unsatisfiable formula)

Φ unsatisfiable means $\Phi \models A$.

Thus (by completeness) there is a Φ-LK proof of A

proof. Thus there is a finite subset Φ' of Φ such that there is a Φ'-LK proof of A

$\therefore$ Φ' is unsatisfiable.

(other direction is easy)

1. The sequent $\to A$ is invalid, Φ = empty

Dealing with Equality

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Definition A *weak $\mathcal{L}$-structure* is an $\mathcal{L}$-structure where $=$ can be any binary predicate

Question: Can we define a finite set of sentences $\mathcal{E}$ that defines equality? (That is, a proper structure satisfies $\mathcal{E}$ and any weak structure satisfying $\mathcal{E}$ must have $=$ be true equality?)

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No! Let $M' = M \cup \{m'\}$ new element

Fix some $m \in M$, and let $m \overset{\mathcal{M}'}{=} m'$

and otherwise $\mathcal{M}'$ on m' behaves like $\mathcal{M}$ on m

Dealing with Equality

Question: Can we define a finite set of sentences $\mathcal{E}$ that defines equality? (That is, a proper structure satisfies $\mathcal{E}$ and any weak structure satisfying $\mathcal{E}$ must have $=$ be true equality?)

But this is the only counterexample.
There is a natural, finite set of axioms that characterizes true equality (up to isomorphism)

Dealing with Equality

Equality Axioms for $\mathcal{L}$ ($\mathcal{E}_{\mathcal{L}}$)

= is an equiv reln

E1. $\forall x (x = x)$

E2. $\forall x \forall y (x = y \supset y = x)$

E3. $\forall x \forall y \forall z ((x = y \wedge y = z) \supset x = z)$

E4. $\forall x_1 .. \forall x_n \forall y_1 .. \forall y_n (x_1 = y_1 \wedge .. \wedge x_n = y_n) \supset f x_1 .. x_n = f y_1 .. y_n$

for all n-ary function symbols, and for all $n \geq 1$

E5. $\forall x_1 .. \forall x_n \forall y_1 .. \forall y_n ((x_1 = y_1 \wedge .. \wedge x_n = y_n) \supset (P x_1 .. x_n \supset P y_1 .. y_n))$

equivalence relation preserved by functions and predicates

Equality Theorem

Theorem Let Φ be a set of $\mathcal{L}$-sentences. Φ is satisfiable iff $\Phi \cup \mathcal{E}_{\mathcal{L}}$ is satisfied by some weak $\mathcal{L}$-structure.

Proof straightforward (see Lecture Notes)

LK with Equality

Add these axioms for all terms $u, t, u_1, \ldots, t_1 \ldots$

L1 $\longrightarrow t = t$

L2 $t = u \longrightarrow u = t$

L3 $t = u,\ u = v \longrightarrow t = v$

L4 $t_1 = u_1, \ldots, t_n = u_n \longrightarrow f t_1 \ldots t_n = f u_1 \ldots u_n$

L5 $t_1 = u_1, \ldots, t_n = u_n,\ P t_1 \ldots t_n \longrightarrow P u_1 \ldots u_n$

Now an LK-Φ proof of $\longrightarrow A$ means an LK proof of A from Φ and from above axioms

Models of $\mathcal{L}_A$

Recall $\mathcal{L}_A = \{0, s, +, \cdot\ ;\ =\}$ Language of arithmetic

the **standard model** for $\mathcal{L}_A$: $\mathbb{N}$

$M = \mathbb{N}$, $0, s, +, \cdot$ have usual meanings

Th(A) or TA: the set of all sentences of $\mathcal{L}_A$ that are true in $\mathbb{N}$

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A nonstandard model of $\mathcal{L}_A$: any model of $\mathcal{L}_A$ that is not isomorphic to the standard model $\mathbb{N}$

Models of $\mathcal{L}_A$

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Defn A set Φ of sentences is **decidable** if there is an algorithm (that always halts) that given a sentence B, outputs 1 if B is in Φ and otherwise outputs 0

We will soon see that TA is **not** decidable.

On the other hand, restricted systems of TA <u>are</u> decidable ($\mathcal{L}_s$, $\mathcal{L}_+$)

Theories

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Definition A theory (over $\mathcal{L}$) is a set Σ of sentences closed under logical consequence. ($\Sigma \models A$ then $A \in \Sigma$)

We can specify a theory by a finite or countable set of sentences Ψ -- the theory corresponding to Ψ is $\{A \mid \Psi \models A\}$

Notation Σ a theory $\Sigma \vdash A$ means $A \in \Sigma$

Definition For a language $\mathcal{L}$, $\Phi_0^{\mathcal{L}}$ is the set of all sentences over $\mathcal{L}$

Theories

Definition Σ is **consistent** if and only if $\Sigma \neq \Phi_0$

(If $\Sigma = \Phi_0$ then Σ contains A & $\neg A$
conversely if Σ contains A & $\neg A$ then
Σ contains all of Φ_0)

Theories

Definición Σ is consistent if and only if $\Sigma \nvdash \Phi_0$

Σ is (negation) complete iff Σ is consistent and for all sentences A, either $\Sigma \vdash A$ or $\Sigma \vdash \neg A$

Theories

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Example $\mathcal{L}_A = \{0, s, +, \cdot\, ; =\}$

TA = all sentences over $\mathcal{L}_A$ that are true in $\mathbb{N}$ is consistent and complete

$\forall x P(x)$, $\forall x\, P(x) \to Q(x)$

$\Longrightarrow \forall x\, Q(x)$

$R(a)$

Theories

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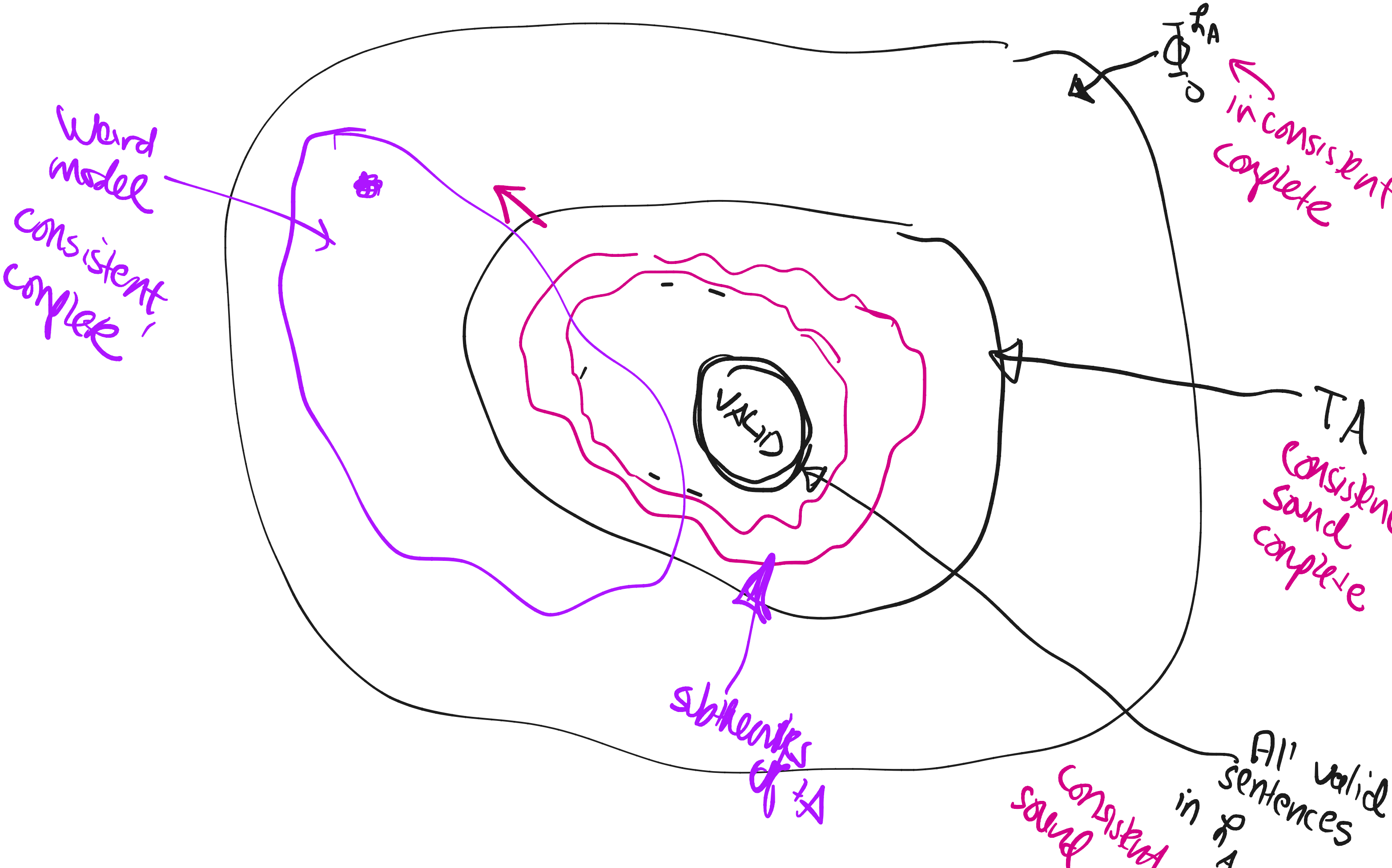
Definición A theory Σ over $\mathcal{L}_A$ is sound (wrt $\mathbb{N}$) iff $\Sigma \subseteq TA$


Word model
consistent complete
$\Phi_0^{\mathcal{L}_A}$
inconsistent complete
VALID
TA
consistent sound complete
subtheories of TA
All valid sentences in $\mathcal{L}_A$
consistent sound

Subsystems of True Arithmetic

- Theory of Successor $(0, s; =)$
- Presburger Arithmetic $(0, s, +; =)$
- Peano Arithmetic $(0, s, +, \cdot\ ; =)$

Defn $\mathcal{L}_s = \{0, s; =\}$ Language of successor

The standard model for $\mathcal{L}_s$, $\mathbb{N}_s$:

$M = \mathbb{N}$, 0 and s have usual meaning $(s(x) = x+1)$

Let $Th(s)$ (theory of successor) be the set of all sentences of $\mathcal{L}_s$ that are true in $\mathbb{N}_s$

The LK completeness theorem stated:

Let $VALID^{\mathcal{L}}$ = set of all sentences that are true in <u>every</u> model

~~LK~~ <u>LK completeness:</u> every $A \in VALID$ has an LK proof

<u>LK soundness:</u> NO $A \notin VALID$ has an LK proof

Incompleteness of PA (peano arithmetic)

Th(S): There is a simple (infinite but countable) complete set of axioms for Th(S), Ψ_S

Ψ_S:

(S1) $\forall x\, (Sx \neq 0)$

(S2) $\forall x \forall y\, (Sx = Sy \supset x = y)$

(S3) $\forall x\, (x = 0 \vee \exists y\, (x = Sy))$

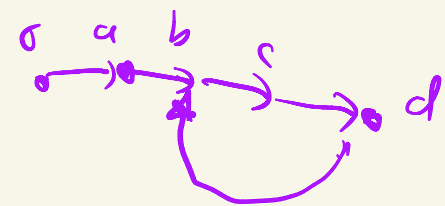
(S4) $\forall x\, (Sx \neq x)$

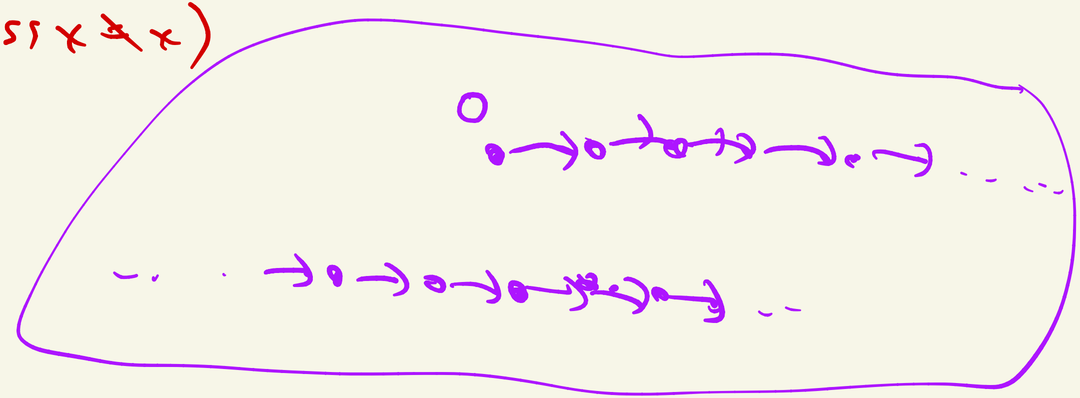
(S5) $\forall x\, (SSx \neq x)$

(S6) $\forall x\, (SSSx \neq x)$

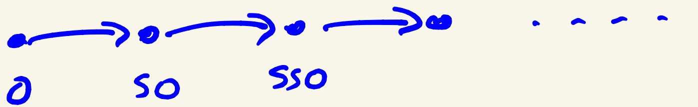
(S7)

$\vdots$





Models for Ψ_s: A model for Ψ_s is a model/structure over $\mathcal{L}_s$ that satisfies all formulas in Ψ_s

(1) 

← isomorphic to $\mathbb{N}$ up to renaming

(2) 

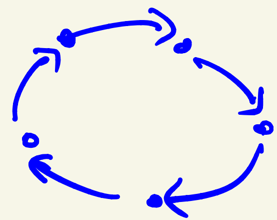
plus

← $\mathbb{N}$ plus a copy of integers

(3) generalizing (2), models contain one copy of $\mathbb{N}$, plus any number of copies (isomorphic to) the integers

Note without all axioms S4, S5, S6, ...
we could have additional models with loops

$$0 \to S0 \to SS0 \to \cdot \quad \cdots$$

plus  any number of cycles

<u>Theorem</u> Ψ_s is complete and consistent

(proof omitted)

Therefore although Ψ_s has both the standard model $\mathbb{N}$ as well as nonstandard models, all models $\mathcal{M}$ of Ψ_s have the <u>same</u> set of true sentences.

Theorem Ψ_s is complete and consistent

(proof omitted)

Therefore although Ψ_s has both the standard model $\mathbb{N}$ as well as nonstandard models, all models $\mathcal{M}$ of Ψ_s have the same set of true sentences.

We'll see later that when a set of sentences (such as $Th(s)$) has a nice (enumerable) axiomatization, then $Th(s)$ is decidable.

Defn $\mathcal{L}_+ = \{0, s, + \,;\, =\}$ Language of Presburger arithmetic

the standard model for $\mathcal{L}_+$, $\underline{\mathbb{N}_+}$:

$M = \mathbb{N}$, $0, s, +$ have usual meaning

$\underline{Th(+)}$: (theory of Presburger arithmetic, or standard model for $\mathcal{L}_+$): all sentences of $\mathcal{L}_+$ that are true in $\underline{\mathbb{N}_+}$

Presburger (1928) showed that $Th(+)$ is also characterized by a countable set of axioms (like the theory of successor)

so it is also consistent and complete

Peano Arithmetic: $\mathcal{L}_A = \{0, S, +, \cdot\,; =\}$

- Has a countable set of axioms
- We think it is consistent
- Has standard model $\mathbb{N}$
  also has not tame nonstandard models

$$A(0) \wedge (\forall x (A(x) \to A(Sx)) \supset \forall x\, A(x)$$

BACK TO TA (TRUE ARITHMETIC)

the standard model for $\mathcal{L}_A$, $\underline{\mathbb{N}}$:

$M = \mathbb{N}$, $0, s, +, \cdot$ have usual meaning

$\underline{Th(A) \text{ or } TA}$: (Theory of True Arithmetic): set of all $\mathcal{L}_A$ sentences that are true in standard model $\underline{\mathbb{N}}$

Theorem TA has a nonstandard model

Theorem TA has a nonstandard model

Proof Let c be a constant symbol (not in $\mathcal{L}_A$)

$$\Psi = \{ c \neq 0,\ c \neq s0,\ c \neq ss0,\ c \neq sss0,\ \ldots \}$$

- every finite subset of Ψ is satisfiable
- so by compactness, $TA \cup \Psi$ has a model $\mathcal{M}$
- $\mathcal{M}$ is not isomorphic to $\underline{\mathbb{N}}$ (standard model) since c cannot be any element of $\mathbb{N}$