Announcements

- HW2 Due MONDAY NOV 15
- Test 2 MONDAY NOV 22

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TODAY:

- ~~One~~ Two more examples showing a Language Not recursive

- Introduction to Incompleteness

I.

$L = \{ x \mid \{x\} \text{ accepts at least one input} \}$

Claim: L is r.e. but not recursive.

1. $\boxed{L \text{ is r.e.}}$ Enumerate all strings in $\{0,1\}^*$

$\{ \varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots \}$

Alg

Dovetail Procedure for L on input x:

For $i = 1, 2, 3, \ldots$

For $j = 1, \ldots i$

Simulate $\{x\}$ on w_j for i steps

If any of the simulations accepts, HALT & accept

$x \in L \Rightarrow$ Alg on x halts & accepts

$x \notin L \Rightarrow$ Alg on x will not halt & therefore won't accept x

$L = \{ x \mid \{x\} \text{ accepts at least one input} \}$

② $\boxed{L \text{ is not recursive}}$

$L_1 = K = \{ y \mid \{y\}(y) \text{ halts} \}$

Assume $L_2 = L$ is recursive & let M_2 be TM $\mathcal{L}(M_2) = L$
and M_2 always halts

M_1 on input y:

Construct encoding z of TM $\{z\}$ where
$\{z\}$ on input x: Ignores x & runs $\{y\}$ on y
and accepts x if $\{y\}(y)$ halts

Run M_2 on z and accept y iff $M_2(z)$ accepts

Claim $\mathcal{L}(M_1) = K$ and M_1 always halts

$y \in K \Rightarrow \{y\}(y)$ halts $\Rightarrow \{z\}$ accepts all inputs $\Rightarrow M_2(z) = 1 \Rightarrow M_1(y) = 1$

$y \notin K \Rightarrow \{y\}(y)$ doesn't halt $\Rightarrow \{z\}$ accepts No input $\Rightarrow M_2(z) \neq 1 \Rightarrow M_1(y) \neq 1$

$L_2 = \{ x \mid \{x\} \text{ accepts at least one input} \}$

$\therefore$ L_2 is r.e. but not recursive

so $\overline{L_2} = \{ x \mid \{x\} \text{ accepts no inputs} \}$
is not r.e.

Say we had a language L not r.e.
& we want to show L' is also not re. Then we would
use an alleged TM M' st. $\mathcal{L}(M') = L'$ in order to construct
a TM M st $\mathcal{L}(M) = L$

II.

$L = \{ x \mid \{x\}$ accepts an even number of inputs in $\{0,1\}^*$ $\}$

$\overline{L} = \{ \langle M \rangle \mid M$ accepts an infinite # of inputs or an odd number of inputs $\}$

<u>Claim</u>

L is not r.e.

Let $\overline{Halt} = \{ \langle x, y \rangle \mid \{x\}$ does <u>Not</u> halt on input $y \}$

$\overline{Halt} = \{ \langle x,y \rangle ; \ \{x\}$ does not halt on $y \}$ ← NOT r.e.

Assume for sake of contradiction that L is r.e.,
& let M be a TM st $\mathcal{L}(M) = L$.

Construct a TM, $M_{\overline{HALT}}$ for $\overline{HALT}$:

$M_{\overline{HALT}}$ on input $\langle x, y \rangle$:

Want: to design $z_{x,y}$ such that :

$\{z_{x,y}\}$ accepts an even number of inputs
iff $\{x\}$ does not halt on y

$\{Z_{xy}\}$ on input w:

if $w = 0$ run $\{x\}$ on y and if $\{x\}$ halts on y accept w

if $w \neq 0$ halt and reject

$M_{\overline{HALT}}$ on input (x, y):

Construct encoding $Z_{x,y}$ of TM $\{Z_{xy}\}$

Run M (TM for L) on Z_{xy}

accept (x, y) iff $M(Z_{xy})$ halts & accepts

$\{Z_{xy}\}$ on input w:

if $w=0$ run $\{x\}$ on y and if $\{x\}$ halts on y accept w

if $w \neq 0$ halt and reject

$M_{\overline{HALT}}$ on input (x,y):

Construct encoding $Z_{x,y}$ of TM $\{Z_{xy}\}$

Run M (TM for L) on Z_{xy}

accept (x,y) iff $M(Z_{xy})$ halts & accepts

<u>Correctness</u>:

1. $\{x\}$ does not halt on input y.
   Then $\{Z_{xy}\}$ accepts no inputs, so $\{Z_{xy}\}$ accepts an even number of inputs. Thus $M(Z_{xy})$ accepts, so $M_{\overline{HALT}}(x,y)$ accepts

2. $\{x\}$ halts on input y.
   then $\{Z_{xy}\}$ accepts only one input $(w=0)$, so $\{Z_{xy}\}$ accepts an odd number of inputs. Thus $M(Z_{xy})$ does not accept so $M_{\overline{HALT}}(x,y)$ does not accept.

Thus we have shown that if L is r.e.
then $\overline{HALT}$ is r.e.

Since $\overline{HALT}$ is <u>not</u> r.e., we have proven that

L is not r.e.

Review of Definitions (p. 75)

$\mathcal{L}_A = \{0, s, +, \cdot\ ; =\}$ Language of arithmetic

Φ_0 = all $\mathcal{L}_A$-sentences

$TA = \{A \in \Phi_0 \mid \mathbb{N} \models A\}$ True Arithmetic

A <u>theory</u> Σ is a set of sentences (over $\mathcal{L}_A$) closed under logical consequence

- We can specify a theory by a subset of sentences that logically implies all sentences in Σ

Σ is <u>consistent</u> iff $\Phi_0 \neq \Sigma$ (iff $\forall A \in \Phi_0$, either A or $\neg A$ Not in Σ)

Σ is <u>complete</u> iff Σ is consistent and $\forall A$ either A or $\neg A$ is in Σ

Σ is sound wrt $\mathfrak{M}$ iff $\Sigma \subseteq TA$

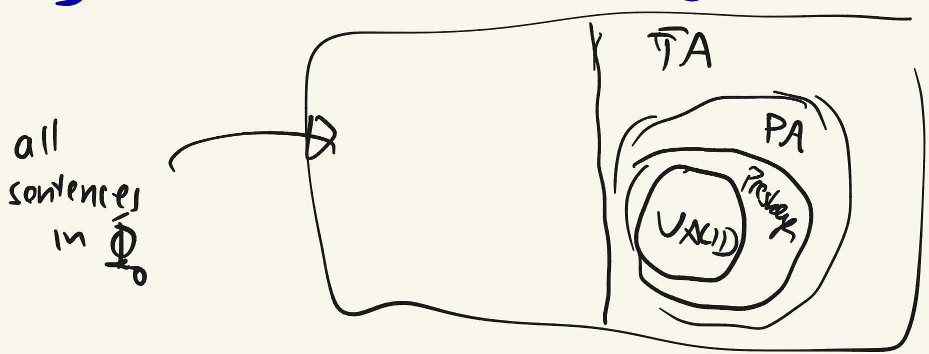
Let $\mathfrak{M}$ be a model/structure over $\mathcal{L}_A$

$Th(\mathfrak{M}) = \{ A \in \Phi_0 \mid \mathfrak{M} \models A \}$

$Th(\mathfrak{M})$ is complete (for all structures $\mathfrak{M}$)

Note $TA = Th(\mathbb{N})$ is complete, consistent, & sound

$VALID = \{ A \in \Phi_0 \mid \models A \}$ ← smallest theory



Let Σ be a theory

Σ is <u>axiomatizable</u> if there exists a set $\Gamma \subseteq \Sigma$

such that ① Γ is recursive

② $\Sigma = \{ A \in \Phi_0 \mid \Gamma \models A \}$

<u>Theorem</u> Σ is axiomatizable iff Σ is r.e.

(p. 76 of Notes)

*** Theorem**

Let $f : \mathbb{N} \to \mathbb{N}$

Let $R_f \subseteq \mathbb{N} \times \mathbb{N}$ be the set of all pairs (x,y) such that $f(x)=y$

then f computable if and only if R_f is r.e.

*** Theorem** A relation $A \subseteq \mathbb{N}$ is r.e.

if and only if there is a recursive relation

$R \subseteq \mathbb{N}^2$ such that

$$x \in A \iff \exists y \, R(x,y) \quad \forall x \in \mathbb{N}$$

Let Σ be a theory

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such that ① Γ is recursive

② $\Sigma = \{ A \in \Phi_0 \mid \Gamma \models A \}$

<u>Theorem</u> Σ is axiomatizable iff Σ is r.e.

<u>Proof</u> $\Rightarrow$. Suppose Σ is axiomatizable, Γ recursive

Define $R(x,y)$ = true iff y encodes a Γ-LK proof of (the formula encoded by) x

R is recursive, so by previous *Theorem, Σ is r.e.

Let Σ be a theory

Σ is <u>axiomatizable</u> if there exists a set $\Gamma \subseteq \Sigma$ such that ① Γ is recursive

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Σ is <u>axiomatizable</u> if there exists a set $\Gamma \subseteq \Sigma$ such that ① Γ is recursive

② $\Sigma = \{ A \in \Phi_0 \mid \Gamma \models A \}$

<u>Theorem</u> Σ is axiomatizable iff Σ is r.e.

<u>Proof</u>

$\Leftarrow$ By *Theorem, Σ = range of total computable function f

$\therefore \quad \Sigma = \{ f(0), f(1), f(2), \dots \}$

That is if A_n is the sentence such that $\# A_n = f(n)$

then $A_0, A_1, A_2, \dots$ is an effective enumeration of Σ

Let $B_n = A_0 \wedge A_1 \wedge \dots \wedge A_n$

Let $\Gamma = \{ B_0, B_1, \dots \}$

this is a set of axioms for Σ and is recursive! (can check if some $F = A_0 \wedge A_1 \wedge \dots \wedge A_j$ for some j)