

Announcements

- HW2 Due MONDAY NOV 15
- Test 2 MONDAY NOV 22

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TODAY:

- Two
• ~~One~~ more examples showing a Language
Not recursive
- Introduction to Incompleteness

(I) $L = \{x \mid \{x\} \text{ accepts at least one input}\}$

Claim L is r.e. but not recursive.

(1) L is r.e. Enumerate all strings in $\{0,1\}^*$

$\{\epsilon, 0, 1, 00, 01, 10, 11, 000, \dots\}$

Alg

Do detail procedure for L on input x :

For $i=1, 2, 3, \dots$

For $j=1, \dots, i$

Simulate $\{x\}$ on w_j for i steps

If any of the simulations accepts, HALT + accept

$x \in L \Rightarrow$ Alg on x halts + accepts

$x \notin L \Rightarrow$ Alg on x will not halt + therefore won't accept x

$L = \{x \mid \{x\} \text{ accepts at least one input}\}$

(2) L is not recursive

$L_1 = K = \{y \mid \{y\}(y) \text{ halts}\}$

Assume $L_2 = L$ is recursive + let M_2 be TM $\mathcal{L}(M_2) = L$
and M_2 always halts

M_1 on input y :

Construct encoding z of TM $\{z\}$ where

$\{z\}$ on input x : Ignores x + runs $\{y\}$ on y
and accepts x if $\{y\}(y)$ halts

Run M_2 on z and accept y iff $M_2(z)$ accepts

Claim $\mathcal{L}(M_1) = K$ and M_1 always halts

$y \in K \Rightarrow \{y\}(y) \text{ halts} \Rightarrow \{z\} \text{ accepts all inputs} \Rightarrow M_2(z) = 1 \Rightarrow M_1(y) = 1$

$y \notin K \Rightarrow \{y\}(y) \text{ doesn't halt} \Rightarrow \{z\} \text{ accepts no input} \Rightarrow M_2(z) \neq 1 \Rightarrow M_1(y) \neq 1$

$L_2 = \{x \mid \{x\} \text{ accepts at least one input}\}$

$\therefore L_2$ is r.e. but not recursive

so $\bar{L}_2 = \{x \mid \{x\} \text{ accepts no inputs}\}$
is not r.e.

Say we had a language L not r.e.

+ we want to show L' is also not r.e. Then we would use an alleged TM M' st. $L(M') = L'$ in order to construct a TM M st. $L(M) = L$

II.

$L = \{x \mid \exists x\}$ accepts an even number
of inputs in $\{0,1\}^*$

$\bar{L} = \{ \langle M \rangle \mid M \text{ accepts an infinite \# of inputs
or an odd number of inputs} \}$

claim

\bar{L} is not r.e.

Let $\overline{\text{Halt}} = \{ \langle x, y \rangle \mid \exists x \exists y \text{ does } \underline{\text{not}} \text{ halt on input } y \}$

$\overline{\text{Halt}} = \{ \langle x, y \rangle; \{x\} \text{ does not halt on } y \}$ ← NOT r.e.

Assume for sake of contradiction that L is r.e.,

+ let M be a TM st $L(M) = L$.

Construct a TM, $M_{\overline{\text{HALT}}}$ for $\overline{\text{HALT}}$:

$M_{\overline{\text{HALT}}}$ on input $\langle x, y \rangle$:

want: to design $Z_{x,y}$ such that:

$\{ Z_{x,y} \}$ accepts an even number of inputs
iff $\{x\}$ does not halt on y

$\{Z_{xy}\}$ on input w :

if $w=0$ run $\{x\}$ on y and if
 $\{x\}$ halts on y accept w

if $w \neq 0$ halt and reject

$M_{\overline{\text{HALT}}}$ on input (x,y) :

Construct encoding Z_{xy} of TM $\{Z_{xy}\}$

Run M (TM for L) on Z_{xy}

accept (x,y) iff $M(Z_{xy})$ halts & accepts

$\{Z_{xy}\}$ on input w :

if $w=0$ run $\{x\}$ on y and if
 $\{x\}$ halts on y accept w

if $w \neq 0$ halt and reject

$M_{\overline{\text{HALT}}}$ on input (x,y) :

Construct encoding Z_{xy} of TM $\{Z_{xy}\}$

Run M (TM for L) on Z_{xy}

accept (x,y) iff $M(Z_{xy})$ halts & accepts

Correctness:

1. $\{x\}$ does not halt on input y .

Then $\{Z_{xy}\}$ accepts no inputs, so $\{Z_{xy}\}$ accepts an even number of inputs. Thus $M(Z_{xy})$ accepts, so $M_{\overline{\text{HALT}}}(x,y)$ accepts

2. $\{x\}$ halts on input y .

Then $\{Z_{xy}\}$ accepts only one input ($w=0$), so $\{Z_{xy}\}$ accepts an odd number of inputs. Thus $M(Z_{xy})$ does not accept so

$M_{\overline{\text{HALT}}}(x,y)$ does not accept.

Thus we have shown that if L is r.e.

then $\overline{\text{HALT}}$ is r.e.

Since $\overline{\text{HALT}}$ is not r.e., we have proven that
 L is not r.e.

Review of Definitions (p. 75)

$\mathcal{L}_A = \{0, s, +, \cdot, =\}$ Language of arithmetic

$\bar{\Phi}_0 =$ all \mathcal{L}_A -sentences

$T_A = \{A \in \bar{\Phi}_0 \mid \mathbb{N} \models A\}$ True Arithmetic

A theory Σ is a set of sentences (over \mathcal{L}_A) closed under logical consequence

- We can specify a theory by a subset of sentences that logically implies all sentences in Σ

Σ is consistent iff $\bar{\Phi}_0 \not\equiv \Sigma$ (iff $\forall A \in \bar{\Phi}_0$, either A or $\neg A$ Not in Σ)

Σ is complete iff Σ is consistent and $\forall A$ either A or $\neg A$ is in Σ

Σ is sound iff $\Sigma \subseteq TA$
wrt \mathcal{M}

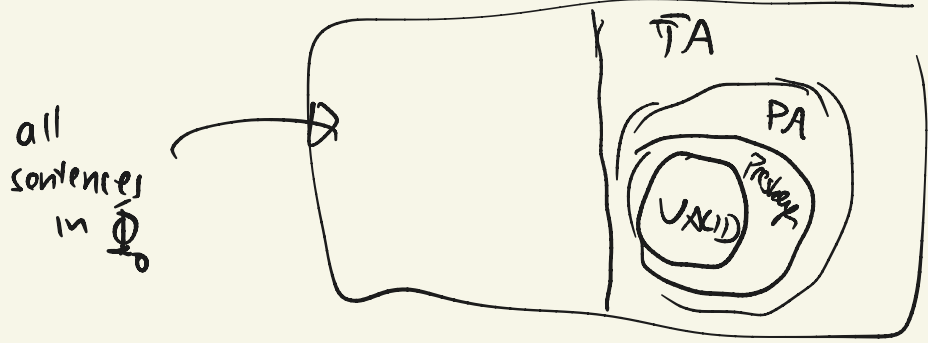
Let \mathcal{M} be a model/structure over \mathcal{L}_A

$$\text{Th}(\mathcal{M}) = \{ A \in \hat{\Phi}_0 \mid \mathcal{M} \models A \}$$

$\text{Th}(\mathcal{M})$ is complete (for all structures \mathcal{M})

Note $TA = \text{Th}(\mathbb{N})$ is complete, consistent, & sound

$$\text{VALID} = \{ A \in \hat{\Phi}_0 \mid \models A \} \leftarrow \text{smallest theory}$$



Let Σ be a theory

Σ is axiomatizable if there exists a set $\Gamma \subseteq \Sigma$

such that ① Γ is recursive

$$\text{② } \Sigma = \{ A \in \mathcal{F}_0 \mid \Gamma \vdash A \}$$

Theorem Σ is axiomatizable iff Σ is r.e.

(p. 76 of Notes)

*Theorem

Let $f: \mathbb{N} \rightarrow \mathbb{N}$

Let $R_f \subseteq \mathbb{N} \times \mathbb{N}$ be the set of all pairs (x, y) such that $f(x) = y$

Then f computable if and only if R_f is r.e.

*Theorem A relation $A \subseteq \mathbb{N}$ is r.e.

if and only if there is a recursive relation

$R \subseteq \mathbb{N}^2$ such that

$$x \in A \iff \exists y R(x, y) \quad \forall x \in \mathbb{N}$$

Let Σ be a theory

Σ is axiomatizable if there exists a set $\Gamma \subseteq \Sigma$

such that ① Γ is recursive

② $\Sigma = \{A \in \mathcal{F}_0 \mid \Gamma \vdash A\}$

Theorem Σ is axiomatizable iff Σ is r.e.

Proof \Rightarrow . Suppose Σ is axiomatizable, Γ recursive

Define $R(x, y) = \text{true}$ iff y encodes a Γ -LK proof
of (the formula encoded by) x

R is recursive, so by previous ***Theorem**, Σ is r.e.

Let Σ be a theory

Σ is axiomatizable if there exists a set $\Gamma \subseteq \Sigma$

such that ① Γ is recursive

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Theorem Σ is axiomatizable iff Σ is r.e.

Proof

\Leftarrow By *Theorem, $\Sigma = \text{range of total computable function } f$
 $\therefore \Sigma = \{f(0), f(1), f(2), \dots\}$

That is if A_n is the sentence such that $\#A_n = f(n)$
then A_0, A_1, A_2, \dots is an effective enumeration of Σ

Let $B_n = A_0 \wedge A_1 \wedge \dots \wedge A_n$

Let $\Gamma = \{B_0, B_1, \dots\}$ \leftarrow

{ this is a set of axioms for Σ
and is recursive!
(can check if some $F = A_0 \wedge A_1 \wedge \dots \wedge A_j$
for some j)