Announcements

- HW1 DUE OCT 6 (8pm)
- Test 1 Wed OCT 13 (in class)

This Week

- Proof Systems for First order Logic

  LK : Soundness and Completeness

FIRST ORDER SEQUENT CALCULUS LK

Lines are again sequents

$$A_1, \ldots, A_k \rightarrow B_1, \ldots, B_\ell$$

where each A_i, B_j is a proper $\mathcal{L}$-Formula

RULES

OLD RULES OF PK

PLUS NEW RULES FOR $\forall$, $\exists$

like a large AND

Large OR

New Logical Rules for $\forall$, $\exists$

$\forall$-left $\quad \dfrac{A(t), \Gamma \rightarrow \Delta}{\forall x\, A(x), \Gamma \rightarrow \Delta}$

$\forall$-Right $\quad \dfrac{\Gamma \rightarrow \Delta, A(b)}{\Gamma \rightarrow \Delta, \forall x\, A(x)}$

$\exists$-left $\quad \dfrac{A(b), \Gamma \rightarrow \Delta}{\exists x\, A(x), \Gamma \rightarrow \Delta}$

$\exists$-right $\quad \dfrac{\Gamma \rightarrow \Delta, A(t)}{\Gamma \rightarrow \Delta, \exists x\, A(x)}$

* A, t are proper
* b is a free variable Not appearing in lower sequent of rule

Example of an LK proof

$$
\text{AND-RT}\;\dfrac{
\exists\text{ Left}\;\dfrac{
\exists\text{-Rt}\;\dfrac{
\text{AND-left}\;\dfrac{
\dfrac{Pa \to Pa}{Pa, Qa \to Pa}
}{Pa \wedge Qa \to Pa}
}{Pa \wedge Qa \to \exists x Px}
}{\exists x (Px \wedge Qx) \to \exists x Px}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{Qa \to Qa}{Pa, Qa \to Qa}
}{Pa \wedge Qa \to Qa}\;\text{AND-Left}
}{Pa \wedge Qa \to \exists x Qx}\;\exists\text{-Rt}
}{\exists x (Px \wedge Qx) \to \exists x Qx}\;\exists\text{-Left}
}{\exists x (Px \wedge Qx) \to \exists x Px \wedge \exists x Qx}
$$

SOUNDNESS

<u>Defn</u> A first order sequent $A_1, \ldots, A_k \to B_1, \ldots, B_\ell$ is **valid** if and only if its associated formula $(A_1 \wedge \ldots \wedge A_k) \supset (B_1 \vee \ldots \vee B_\ell)$ is valid.

Soundness Theorem for LK Every sequent provable in LK is valid

Soundness (Proof) : By induction on the number of sequents in proof

Key Lemma For all rules/axioms of LK

For all instantiations of a rule/axiom

if upper sequents of the rule are valid

then lower sequent is also valid

$$\frac{\Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, A \vee B} \qquad A \rightarrow A$$

Key Lemma (Proof sketch)

Example: $\forall$ Right Rule $\left(\dfrac{\Gamma \to A(b), \Delta}{\Gamma \to \forall x\, A(x), \Delta} \right)$

Assume: $\Gamma \to A(b), \Delta$ is valid. Show $\Gamma \to \forall x\, A(x), \Delta$ is also valid

Let $\Gamma = B_1, \ldots, B_k$, $\Delta = C_1, \ldots, C_\ell$

Then $\underbrace{\neg B_1 \vee \neg B_2 \vee \ldots \vee \neg B_k \vee C_1 \vee \ldots \vee C_\ell}_{D} \vee A(b)$ is valid

ie $\forall \mathfrak{M}, \sigma : \mathfrak{M} \models D \vee A(b)\, [\sigma]$

Show $\forall \mathfrak{M}, \sigma : \mathfrak{M} \models D \vee \forall x\, A(x)\, [\sigma]$

Key Lemma (Proof sketch)

Example: $\forall$Right Rule $\left(\dfrac{\Gamma \to A(b), \Delta}{\Gamma \to \forall x\, A(x), \Delta} \right)$

Assume: $\Gamma \to A(b), \Delta$ is valid. Show $\Gamma \to \forall x\, A(x), \Delta$ is also valid

Let $\Gamma = B_1, \dots, B_k$, $\Delta = C_1, \dots, C_l$

Then $\underbrace{\neg B_1 \vee \neg B_2 \vee \dots \vee \neg B_k \vee C_1 \vee \dots \vee C_l}_{D} \vee A(b)$ is valid

ie $\forall \mathfrak{M}, \sigma : \mathfrak{M} \models D \vee A(b)[\sigma]$

Let σ' be an object assignment to all free variables in $D \vee A(b)$ except for b

Case 1: $\mathfrak{M} \models D[\sigma']$

$\therefore \mathfrak{M} \models D \vee A(b)[\sigma]$ $\forall \sigma$ extending σ' (since b does not occur in D)

Case 2 $\mathfrak{M} \not\models D[\sigma']$.

$\therefore \mathfrak{M} \models A(b)[\sigma', m/b]$ $\forall m \in M$ (since $D \vee A(b)$ is valid)

$\therefore \mathfrak{M} \models A(b)$ $\forall \sigma$ extending σ'

$\therefore \mathfrak{M} \models D \vee A(b)[\sigma]$

TODAY: gödels COMPLETENESS THEOREM

Defn An LK-Φ proof is an LK-proof, but leaves are either axioms ($A \Rightarrow A$) or of the form $\Rightarrow A$ for $A \in \Phi$

goal prove that if $\Gamma \Rightarrow \Delta$ is a logical consequence of Φ, then there is an LK-Φ proof of $\Gamma \Rightarrow \Delta$
(called Derivational completeness)

Defn Let $A(a_1 \ldots a_n)$ be a formula with free variables $a_1 \ldots a_n$. Then $\forall A$ is $\forall x_1 \forall x_2 \ldots \forall x_n A(x_1 \ldots x_n)$
(called universal closure of A)

TODAY: LK COMPLETENESS

(MAIN LEMMA) Completeness Lemma

If $\Gamma \to \Delta$ is a logical consequence of a set of (possibly infinite) formulas $\forall \Phi$ then there exists a finite subset $\{C_1, \ldots, C_n\}$ of Φ such that

$$\forall C_1, \ldots, \forall C_n, \Gamma \to \Delta \text{ has a (cut-free) PK proof}$$

* We will assume = Not in Language for Now

Derivational Completeness Theorem

Let Φ be a set of sequents or formulas such that the sequent $\Gamma \to \Delta$ is a logical consequence of $\forall\Phi$.

Then there is an LK-Φ proof of $\Gamma \to \Delta$.

$\Uparrow$

Proof follows from Completeness Lemma (similar to derivational completeness of PK from completeness)

Proof of LK Completeness Lemma

High Level idea (assume Φ is empty for now)

- As in PK completeness, we want to construct an LK proof in reverse.
- Start with $\Gamma \Rightarrow \Delta$ at root, and apply rules in reverse (to break up a formula into one or 2 smaller ones)
- Tricky rules are $\exists$right & $\forall$left. When applying one of these in reverse, Need to "guess" a term

New Logical Rules for $\forall$, $\exists$

$\forall$-left
$$\frac{A(t), \Gamma \to \Delta}{\forall x\, A(x), \Gamma \to \Delta}$$

$\forall$-Right
$$\frac{\Gamma \to \Delta, A(b)}{\Gamma \to \Delta, \forall x\, A(x)}$$

$\exists$-left
$$\frac{A(b), \Gamma \to \Delta}{\exists x\, A(x), \Gamma \to \Delta}$$

$\exists$-right
$$\frac{\Gamma \to \Delta, A(t)}{\Gamma \to \Delta, \exists x\, A(x)}$$

* A, t are proper
* b is a free variable not appearing in lower sequent of rule

Proof of LK Completeness Lemma

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- Start with $\Gamma \Rightarrow \Delta$ at root, and apply rules in reverse (to break up a formula into one or 2 smaller ones)
- Tricky rules are $\exists$right & $\forall$left.
  When applying one of these in reverse, Need to "guess" a term
- Key is to systematically try all possible terms — without going down a rabbit hole.

Example of an LK proof

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{Pa \to Pa}{Pa,\, Qa \to Pa}}{?\quad Pa \land Qa \to Pa}}{Pa \land Qa \to \exists x Px}}{\exists x(Px \land Qx) \to \exists x Px}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{Qa \to Qa}{Pa,\, Qa \to Qa}}{Pa \land Qa \to Qa \quad ?}}{Pa \land Qa \to \exists x Qx}}{\exists x(Px \land Qx) \to \exists x Qx}
}{
\exists x(Px \land Qx) \to \exists x Px \land \exists x Qx
}
$$

Example of an LK proof

$$Pa, Qa \longrightarrow Pb$$

$$\dfrac{\dfrac{Pa \wedge Qa \longrightarrow Pb}{Pa \wedge Qa \longrightarrow \exists x Px}}{\exists x (Px \wedge Qx) \longrightarrow \exists x Px} \qquad \dfrac{Pa \wedge Qa \longrightarrow \exists x Qx}{\exists x (Px \wedge Qx) \longrightarrow \exists x Qx}$$

$$\overline{\exists x (Px \wedge Qx) \longrightarrow \exists x Px \wedge \exists x Qx}$$

Instead:

$$Pa, Qa \Rightarrow Pb, \exists x Px$$

$$Pa \land Qa \Rightarrow Pb, \exists x Px$$

$$\dfrac{\dfrac{Pa \land Qa \Rightarrow \exists x Px}{\exists x (Px \land Qx) \Rightarrow \exists x Px} \qquad \dfrac{Pa \land Qa \Rightarrow \exists x Qx}{\exists x (Px \land Qx) \Rightarrow \exists x Qx}}{\exists x (Px \land Qx) \Rightarrow \exists x Px \land \exists x Qx}$$

Instead

Try again

$$Pa, Qa \rightarrow Pb, Pfa, \exists x Px$$

$$Pa, Qa \rightarrow Pb, \exists x Px$$

$$Pa \wedge Qa \rightarrow Pb, \exists x Px$$

$$\dfrac{\dfrac{Pa \wedge Qa \rightarrow \exists x Px}{\exists x (Px \wedge Qx) \rightarrow \exists x Px} \qquad \dfrac{Pa \wedge Qa \rightarrow \exists x Qx}{\exists x (Px \wedge Qx) \rightarrow \exists x Qx}}{\exists x (Px \wedge Qx) \rightarrow \exists x Px \wedge \exists x Qx}$$

Instead

$$\vdots$$

and again

$$Pa, Qa \to Pb, Pfa, Pfb, \exists x Px$$

and again

$$Pa, Qa \to Pb, Pfa, \exists x Px$$

Try again

$$Pa, Qa \to Pb, \exists x Px$$

$$Pa \wedge Qa \to Pb, \exists x Px$$

$$\dfrac{\dfrac{Pa \wedge Qa \to \exists x Px}{\exists x (Px \wedge Qx) \to \exists x Px} \qquad \dfrac{Pa \wedge Qa \to \exists x Qx}{\exists x (Px \wedge Qx) \to \exists x Qx}}{\exists x (Px \wedge Qx) \to \exists x Px \wedge \exists x Qx}$$

Instead

There are infinitely many choices! Need a systematic way to try all

and again

and again

$$Pa, Qa \to Pb, Pfa, Pfb, \exists x Px$$

$$Pa, Qa \to Pb, Pfa, \exists x Px$$

Try again

$$Pa, Qa \to Pb, \exists x Px$$

$$Pa \wedge Qa \to Pb, \exists x Px$$

$$\frac{Pa \wedge Qa \to \exists x Px}{\exists x(Px \wedge Qx) \to \exists x Px} \qquad \frac{Pa \wedge Qa \to \exists x Qx}{\exists x(Px \wedge Qx) \to \exists x Qx}$$

$$\exists x(Px \wedge Qx) \to \exists x Px \wedge \exists x Qx$$

Instead

and again

and again

$Pa, Qa \to Pb, Pfa, Pfb, \exists x Px$

Try again

$Pa, Qa \to Pb, Pfa, \exists x Px$

$Pa, Qa \to Pb, \exists x Px$

$Pa \land Qa \to Pb, \exists x Px$

$$\dfrac{\dfrac{Pa \land Qa \to \exists x Px}{\exists x (Px \land Qx) \to \exists x Px} \qquad \dfrac{Pa \land Qa \to \exists x Qx}{\exists x (Px \land Qx) \to \exists x Qx}}{\exists x (Px \land Qx) \to \exists x Px \land \exists x Qx}$$

There are infinitely many choices! Need a systematic way to try all and for all frontier sequents in current proof!

Completeness: Proof Search Algorithm

Enumeration of formulas + terms:

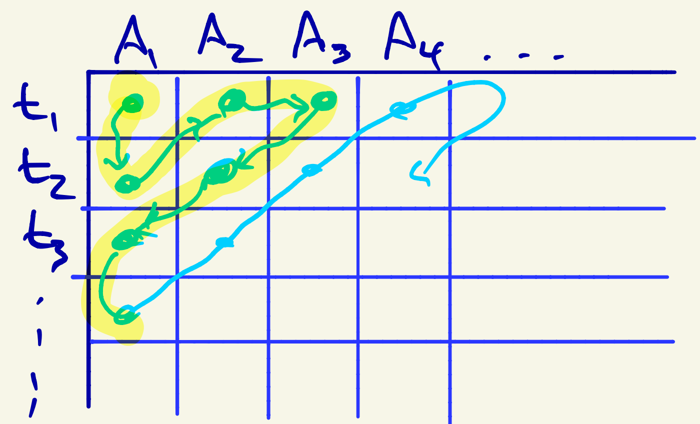
Since the number of underlying symbols of $\mathcal{L}$ is finite, there is an <u>enumeration</u> of pairs $\langle A_1, t_1 \rangle, \langle A_2, t_2 \rangle, \langle A_3, t_3 \rangle, \ldots$ such that <u>every</u> term and <u>every</u> formula in $\mathcal{L}$ occur infinitely often in the enumeration

More details of enumeration ($\mathcal{L}$ finite)

Enumerate all $\mathcal{L}$-formulas $A_1, A_2, \ldots$
Enumerate " $\mathcal{L}$-terms $t_1, \ldots$

such that every formula/term occurs infinitely often

Enumerate all pairs to have same property



Completeness: Proof Search Algorithm

Start with Φ = set of sequents/formulas, $\Gamma \to \Delta$

Want an algorithm that will output an Φ-LK proof of $\Gamma \to \Delta$ whenever $\Phi \models \Gamma \to \Delta$

- Initially Π is the sequent $\Gamma \to \Delta$
- At each stage, modify Π by adding some $A_i \in \Phi$ to antecedent of <u>all</u> sequents in Π, and adding onto the "frontier" or "active" sequents in Π.
- Active sequent: a leaf sequent in Π, not a weakening of $A \to A$
- at stage k: we will use the k^{th} pair $\langle A_k, t_k \rangle$ in the enumeration

Completeness: Proof Search Algorithm

Stage k: $\langle A, t \rangle_k$

(1) If $A_k \in \Phi$, replace $\Gamma' \Rightarrow \Delta'$ in Π by $\Gamma', A_k \Rightarrow \Delta'$

(2) If A_k atomic, skip this step. Otherwise for all leaf sequents containing A_k, break up outermost connective of A_k using the appropriate logical rule, and t_k if necessary.

Completeness: Proof Search Algorithm

Stage k:

(1) If $A_k \in \Phi$, replace $\Gamma' \to \Delta'$ in Π by $\Gamma', A_k \to \Delta'$

(2) If A_k atomic, skip this step. Otherwise for all leaf sequents containing A_k, break up outermost connective of A_k using the appropriate logical rule, and t_k if necessary.

Examples:

- $A_k = \exists x B x$

$$\frac{\Gamma, B(c) \to \Delta}{\Gamma, \exists x B(x) \to \Delta}$$

c is a new variable

$$\frac{\Gamma \to \Delta, \exists x B(x), B(t_k)}{\Gamma \to \Delta, \exists x B(x)}$$

keep both here

Completeness: Proof Search Algorithm

<u>Stage k</u>:

(1) If $A_k \in \Phi$, replace $\Gamma' \to \Delta'$ in Π by $\Gamma', A_k \to \Delta'$

(2) If A_k atomic, skip this step. Otherwise for all leaf sequents containing A_k, break up outermost connective of A_k using the appropriate logical rule, and t_k if necessary.

Examples:

- $A_k = \forall x B(x)$

$$\frac{\Gamma \to \Delta, B(c)}{\Gamma \to \Delta, \forall x B(x)}$$

c a new variable

$$\frac{B(t_k), \forall x B(x), \Gamma \to \Delta}{\Gamma, \forall x B(x) \to \Delta}$$

keep both here

<u>Exit</u> when no more active sequents

Proof of Correctness

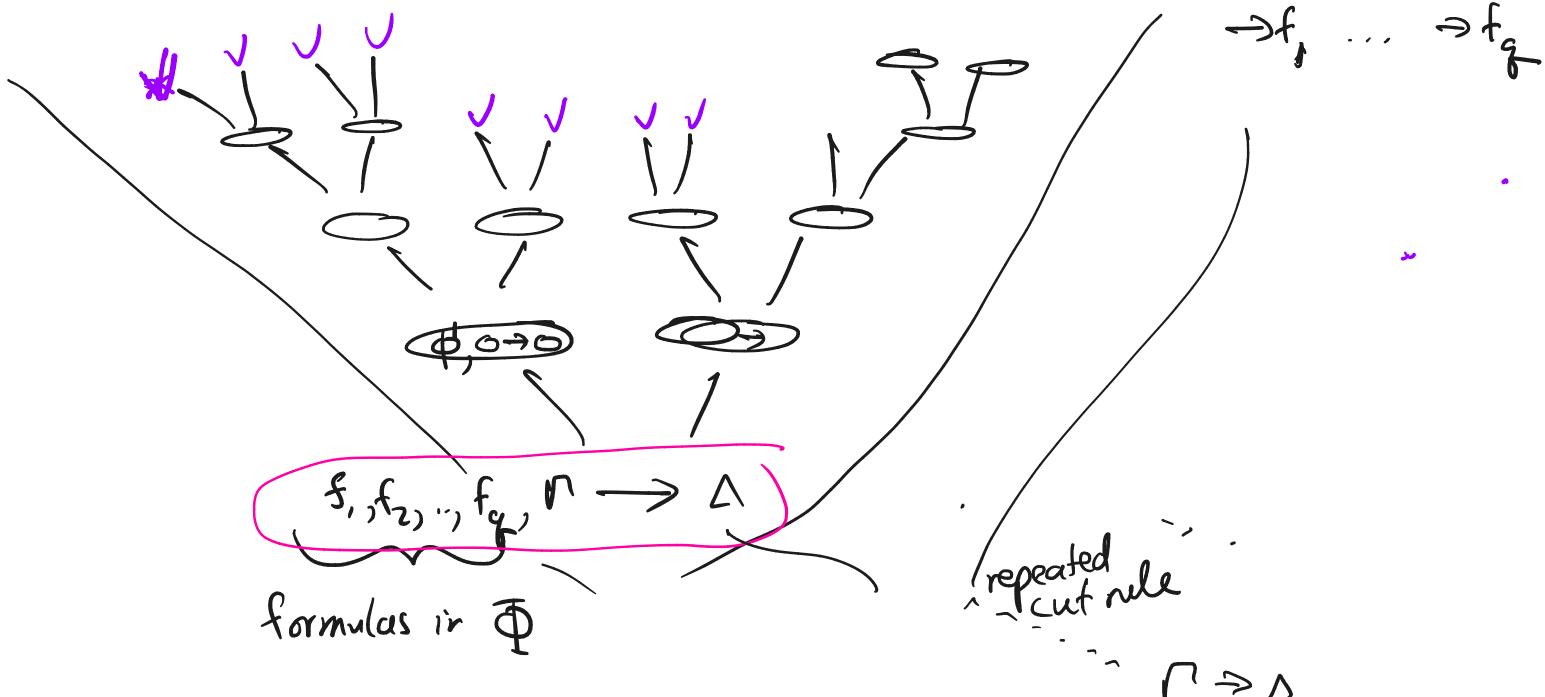
We want to show:

- If Algorithm halts, Π is an LK-Φ proof of $\Gamma \to \Delta$ ✓

- If Algorithm never halts, then $\forall \Phi \not\models \Gamma \to \Delta$

Show: If our algorithm halts when run on Φ, $\Gamma \to \Delta$ then it produces a Φ-LK "proof" of $\Gamma \to \Delta$

What will proof tree look like if Alg halts?



Proof of Correctness

We want to show: If Algorithm never halts, then $\Phi \nvDash \Gamma \to \Delta$
(or if it halts but some leaf sequent doesn't contain some formula A on both left & rt of $\to$)

Suppose Algorithm doesn't halt and let Π be the (typically infinite) tree that results

Leaf "sequents" of Π look like $\Gamma', \underbrace{C_1, C_2, \ldots}_{\text{infinite sequence containing all of } \Phi \text{ each infinitely often}} \to \Delta'$

Find a bad path β in the tree:
If Π finite, $\exists$ some active leaf node containing only atomic formulas. Choose β to be path from root to this leaf

Proof of Correctness

We want to show: If Algorithm never halts, then $\forall \Phi \not\models \Gamma \to \Delta$

Find a bad path β in the tree:

If Π finite, $\exists$ some active leaf node containing only atomic formulas. Choose β to be path from root to this leaf

If Π infinite by König's Lemma, $\exists$ an infinite path. Let β be this path

Proof of Correctness

Properties of β

(1) β is a path starting at root
(2) all sequents in β were once active
(3) for all sequents in β, no formula occurs on both the Left and right side of sequent
(4) all atomic formulas $A \in \Phi$ in root sequent of β on LEFT, and thus occur on LEFT of all sequents in β

By (3) + (4), we know that NO atomic $A \in \Phi$ occurs on the Right of any sequent in β

Proof of Correctness (cont'd)

We will construct a "term" model $\mathfrak{M}$, & object assignment σ from β such that $\mathfrak{M} \models \Phi[\sigma]$ but $\mathfrak{M} \not\models \Gamma \to \Delta$ (and thus our algorithm fails to halt & produce a proof only when $\Gamma \to \Delta$ is not a logical consequence of Φ.)

Let M (universe) be all terms over $\mathcal{L}$

Nice thing is then our interpretation of all functions is natural: If we have a term

f: pairs of universe elements $\to$ universe element

$$f(\overline{S1}, \overline{SSS1}) = \overline{\underbrace{fS1SSS1}}$$

Proof of Correctness (cont'd)

We will construct a "term" model $\mathfrak{M}$, + object assignment σ from β such that $\mathfrak{M} \models \Phi[\sigma]$ but $\mathfrak{M} \not\models \Gamma \to \Delta$

Universe M: all $\mathcal{L}$-terms t (containing only free vars)

σ : map variable a to itself ($\sigma(a) = \bar{a}$)

$$f^{\mathfrak{M}}(\bar{r}_1 \ldots \bar{r}_k) \stackrel{d}{=} \overline{f r_1 \ldots r_k}$$

$$P^{\mathfrak{M}}(r_1 \ldots r_k) \stackrel{d}{=} \text{true if and only if } P r_1 \ldots r_k$$

is on the LEFT of some sequent in β

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathcal{M}, \sigma$ satisfies A iff A is on the LEFT of some sequent in β, and

$\mathcal{M}, \sigma$ falsifies A iff A is on the RIGHT of some sequent in β

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathfrak{M}, \sigma$ satisfies A iff A is on the LEFT of some sequent in β, and

$\mathfrak{M}, \sigma$ falsifies A iff A is on the RIGHT of some sequent in β

Proof (induction on A)

A atomic: A cannot occur on LEFT of some sequent in β and on RIGHT of some sequent in β (since A persists up β)

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathcal{M}, \sigma$ satisfies A iff A is on the LEFT of some sequent in β, and

$\mathcal{M}, \sigma$ falsifies A iff A is on the RIGHT of some sequent in β

Proof (induction on A)

Induction Step Example $A = \exists x\, B(x)$ on RIGHT

high level: if A occurs in some sequent in β, then A persists upward until it becomes the active formula (at stage k, $A_k = A$) then use inductive hypothesis

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathcal{M}, \sigma$ satisfies A iff A is on the LEFT of some sequent in β, and

$\mathcal{M}, \sigma$ falsifies A iff A is on the RIGHT of some sequent in β

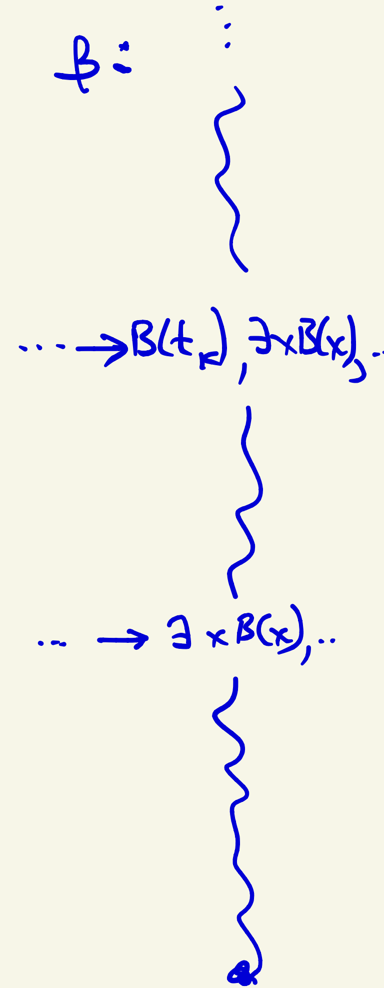
Proof (induction on A)

Induction Step $A = \exists x B(x)$ on RIGHT

By Ind hyp, M, σ falsify $B(t_j)$

Since $\exists x B(x)$ persists, we have $\forall t$ $B(t)$ on RIGHT of some sequent in β

Thus $\mathcal{M}, \sigma$ falsify $B(t)$ for all terms t



Test on wednesday

~4 questions

- → Satisfiable, Valid/Tautology, UNSAT] short answer
- → Proof systems Resolution + PK
  - completeness for both
  - Implicational completeness for PK
- → compactness]
- → key Definitions, Key Theorems] short answer