Announcements

- HW1 DUE OCT 6 (8pm)
- Test 1 Wed OCT 13 (in class)

This Week

- First Order Logic

  Language / Syntax

  Semantics : Models

  (pages 18 - 27 of course notes)

  last class

- Proof Systems for First order Logic

  today & Next week

FIRST ORDER LOGIC

Underlying language $\mathcal{L}$ specified by:

1. $\forall n \in \mathbb{N}$ a set of n-ary function symbols (i.e.: $f, g, h, +, \cdot$)

   0-ary function symbols are called constants

2. $\forall n \in \mathbb{N}$ a set of n-ary predicate symbols (i.e. $P, Q, R, <, \leq$)

Plus:

- Variables: $x, y, z, \ldots a, b, c, \ldots$
- $\neg, \vee, \wedge, \exists, \forall$
- parenthesis $($, $)$

$\Big\}$ Built in symbols

Example $\mathcal{L}_A$ (language of arithmetic)

$$\mathcal{L}_A = \{ \underbrace{0, s, +, \cdot}_{\text{function symbols}} \; ; \underbrace{=}_{\text{relation symbols}} \}$$

0	constant (0-ary function symbol)
s	unary function symbol
$+, \cdot$	binary function symbols
$=$	binary predicate symbol

Terms over $\mathcal{L}$

(1) Every variable is a term

(2) If f is an n-ary function symbol, and $t_1, \ldots, t_n$ terms, then $f t_1 \ldots t_n$ is a term

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Examples of terms

$(0, s, f, +, \cdot)$

0-ary: 0; unary: s; binary: $f, +, \cdot$

$f0sss0$, $+xfyz$, $\cdot + ab\, ss0$

$f(0, sss0)$ $\quad$ $x + f(y, z)$ $\quad$ $(a+b) \cdot ss0$

FIRST ORDER FORMULAS OVER $\mathcal{L}$

(1) $Pt_1 \ldots t_n$ is an atomic $\mathcal{L}$-formula, where P is an n-ary predicate in $\mathcal{L}$, and $t_1 \ldots t_n$ are terms over $\mathcal{L}$

(2) If A, B are $\mathcal{L}$-formulas, so are
$\neg A$, $(A \wedge B)$, $(A \vee B)$, $\forall x A$, $\exists x A$

FREE / BOUND VARIABLES

- An occurrence of x in A is **bound** if x is in a subformula of A of the form $\forall x B$, or $\exists x B$ (otherwise x is **free** in A)

<u>Example</u> $\exists y (x = y + y)$

$Px \wedge \forall x (\neg(x + Sx = x))$

- A formula/term is **closed** if it contains no free variables
- A closed formula is called a **sentence**

SEMANTICS OF FO LOGIC

An $\mathcal{L}$-structure $\mathfrak{M}$ (or model) consists of:

① A nonempty set M called the universe
(variables range over M)

② For every n-ary function symbol f in $\mathcal{L}$, an associated function $f^{\mathfrak{M}} : M^n \to M$

③ For each n-ary relation symbol P in $\mathcal{L}$, an associated relation $P^{\mathfrak{M}} \subseteq M^n$

* Equality predicate $=$ is always true equality relation on M.

Examples

$$\mathcal{L}_A = \{0, s, +, \cdot \,; =\}$$

① $\mathcal{M} = \underline{\mathbb{N}}$,

$0 = 0 \in \mathbb{N}$

s: successor. ie. $s(2) = 3$, ...

$+$: plus. ie, $+(0,1) = 1$, $+(2,3) = 5$, etc

$\cdot$: times

② $M = \{$ ■, ●, ★ $\}$ $\quad 0 =$ ■

s(■) = ●

s(●) = ■

s(★) = ★

+	■	●	★
■	●	●	★
●	●	●	★
★	★	★	★

•	■	●	★
■	■	★	●
●	■	■	★
★	★	★	■

Definition: Evaluation of terms/formulas over $\mathfrak{M}, \sigma$

Let $\mathfrak{M}$ be an $\mathcal{L}$-structure,
σ an object assignment for $\mathfrak{M}$

Evaluation of terms over $\mathfrak{M}, \sigma$

(a) $x^{\mathfrak{M}}[\sigma]$ is $\sigma(x)$ for all variables x

(b) $(f t_1 .. t_n)^{\mathfrak{M}}[\sigma] = f^{\mathfrak{M}}(t_1^{\mathfrak{M}}[\sigma], \ldots, t_n^{\mathfrak{M}}[\sigma])$

Example $\sigma: x_1 \to 5 \quad x_2 \to 7$

$S(x_1 + x_2)[\sigma] = 13$

Evaluation of formulas over $\mathfrak{M}, \sigma$

Let A be an $\mathcal{L}$-formula. $\mathfrak{M} \models A[\sigma]$ ($\mathfrak{M}$ satisfies A under σ) iff

(a) $\mathfrak{M} \models Pt_1 \dots t_n[\sigma]$ iff $\langle t_1^{\mathfrak{M}}[\sigma], \dots, t_n^{\mathfrak{M}}[\sigma] \rangle \in P^{\mathfrak{M}}$

(b) $\mathfrak{M} \models (s = t)[\sigma]$ iff $s^{\mathfrak{M}}[\sigma] = t^{\mathfrak{M}}[\sigma]$

(c) $\mathfrak{M} \models \neg A[\sigma]$ iff Not $\mathfrak{M} \models A[\sigma]$

(d) $\mathfrak{M} \models (A \vee B)[\sigma]$ iff $\mathfrak{M} \models A[\sigma]$ or $\mathfrak{M} \models B[\sigma]$

(e) $\mathfrak{M} \models (A \wedge B)[\sigma]$ iff $\mathfrak{M} \models A[\sigma]$ and $\mathfrak{M} \models B(\sigma)$

(f) $\mathfrak{M} \models \forall x A[\sigma]$ iff $\forall m \in M$ $\mathfrak{M} \models A[\sigma(m/x)]$

(g) $\mathfrak{M} \models \exists x A[\sigma]$ iff $\exists m \in M$ $\mathfrak{M} \models A[\sigma(m/x)]$

Example

$\mathcal{L}_A = \{0, s, +, \cdot\ ;\ =\}$

$\mathfrak{M}$: $M = \mathbb{N}$, $0, s, +, \cdot$ have usual interpretations

A : $\forall x \forall y (x + y = z)$ B : $\forall x \forall y (x + sy = s(x+y))$

σ : $x = 2$ $y = 3$ $z = 7$

$\mathfrak{M} \cancel{\overset{?}{\models}} A[\sigma]$

$\mathfrak{M} \overset{?}{\models} B[\sigma]$

IMPORTANT DEFINITIONS

Let A be a f.o. formula over $\mathcal{L}$.

① A is satisfiable iff there exists a model $\mathcal{M}$ and an object assignment σ such that $\mathcal{M} \models A[\sigma]$

② A set of formulas Φ is satisfiable iff $\exists \mathcal{M}, \sigma$ such that $\mathcal{M} \models \Phi[\sigma]$ [$\mathcal{M} \models A[\sigma]$ for all $A \in \Phi$]

③ $\Phi \models A$ (A is a logical consequence of Φ) iff $\forall \mathcal{M}\ \forall \sigma$ if $\mathcal{M} \models \Phi[\sigma]$ then $\mathcal{M} \models A[\sigma]$

$\models A$ (A is valid) iff $\forall \mathcal{M}, \sigma$ $\mathcal{M} \models A[\sigma]$

④ $A \Leftrightarrow B$ (A and B are logically equivalent)

iff $\forall \mathfrak{M}\ \forall \sigma \quad \mathfrak{M} \models A[\sigma]$ iff $\mathfrak{M} \models B[\sigma]$

Substitution

Let s,t be $\mathcal{L}$-terms.

$t(s/x)$: substitute x everywhere by s

$A(s/x)$: substitute all free occurrences of x in A by s

For readability, we will write

$t = +\,SSO\,x$ as $SSO + x$

Substitution

Let s, t be $\mathcal{L}$-terms.

$t(s/x)$: substitute x everywhere by s

$A(s/x)$: substitute all free occurrences of x in A by s

Lemma $$(t(s/x))^{\mathcal{M}}[\sigma] = t^{\mathcal{M}}\left[\sigma\left(\frac{s^{\mathcal{M}}[\sigma]}{x}\right)\right]$$

substitute x for s to get t' then evaluate t' under $\mathcal{M}, \sigma$

obtain new object assignment σ' where $\sigma'(x) = s^{\mathcal{M}}$ then evaluate t under $\mathcal{M}, \sigma'$

Substitution Cont'd

Need to be more careful when making substitutions into formulas

<u>Example</u>: $A : \forall y \neg (x = y + y)$

$$A\left(\frac{x+y}{x}\right) : \forall y \neg (x+y = y+y)$$

<u>Defn</u> term t is <u>freely substitutable</u> for x in A iff there is no subformula in A of the form $\forall y B$ or $\exists y B$ where y occurs in t

Substitution Theorem

If t is freely substitutable for x in A

then $\forall \mathfrak{M}\ \forall \sigma$

$$\mathfrak{M} \models A(t/x)[\sigma] \text{ iff } \mathfrak{M} \models A\left[\sigma\left(t^{\mathfrak{M}}[\sigma]/x\right)\right]$$

Easy way to avoid this problem
(of making a "bad" substitution):

2 types of variables
free variables $a, b, c, \ldots$

bound variables $x, y, z, \ldots$

Proper formula : every free variable occurrence is of type free & every bound variable occurrence of type bound

Proper term : No variables of type bound

FIRST ORDER SEQUENT CALCULUS LK

Lines are again sequents

$$\left. A_1, \ldots, A_k \rightarrow B_1, \ldots, B_\ell \right\} S$$

where each A_i, B_j is a proper $\mathcal{L}$-formula

$$A_S : \quad A_1 \wedge A_2 \wedge \ldots \wedge A_k \supset B_1 \vee \ldots \vee B_\ell$$

FIRST ORDER SEQUENT CALCULUS LK

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RULES

OLD RULES OF PK

PLUS NEW RULES FOR $\forall$, $\exists$

like a large AND

Large OR

New Logical Rules for $\forall$, $\exists$

$\forall$-left
$$\frac{A(t), \Gamma \to \Delta}{\forall x A(x), \Gamma \to \Delta}$$

$\forall$-Right
$$\frac{\Gamma \to \Delta, A(b)}{\Gamma \to \Delta, \forall x A(x)}$$

$\exists$-left
$$\frac{A(b), \Gamma \to \Delta}{\exists x A(x), \Gamma \to \Delta}$$

$\exists$-right
$$\frac{\Gamma \to \Delta, A(t)}{\Gamma \to \Delta, \exists x A(x)}$$

* A, t are proper
* b is a free variable Not appearing in lower sequent of rule

Example of an LK proof

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{Pa \to Pa}{Pa, Qa \to Pa}}{Pa \wedge Qa \to Pa}\;\text{AND-left}}{Pa \wedge Qa \to \exists x Px}\;\exists\text{-Rt}}{\exists x (Px \wedge Qx) \to \exists x Px}\;\exists\text{ Left}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{Qa \to Qa}{Pa, Qa \to Qa}}{Pa \wedge Qa \to Qa}\;\text{AND-Left}}{Pa \wedge Qa \to \exists x Qx}\;\exists\text{-Rt}}{\exists x (Px \wedge Qx) \to \exists x Qx}\;\exists\text{-Left}
}{\exists x (Px \wedge Qx) \to \exists x Px \wedge \exists x Qx}\;\text{AND-RT}
$$

SOUNDNESS

Defn A first order sequent $A_1, \ldots, A_k \rightarrow B_1, \ldots, B_\ell$ is valid if and only if its associated formula $(A_1 \wedge \ldots \wedge A_k) \supset (B_1 \vee \ldots \vee B_\ell)$ is valid.

Soundness Theorem for LK Every sequent provable in LK is valid

Proof of Lemma

go through each rule

Example: $\forall$-right rule

$$\frac{\Gamma \to \Delta, A(a)}{\Gamma \to \Delta, \forall x\, A(x)} \qquad \begin{matrix} \leftarrow S \\ \leftarrow S' \end{matrix}$$

Note: a cannot occur in lower sequent & thus a can't occur in any B_i, C_j

Let $\Gamma = B_1 \dots B_\ell$
$\Delta = C_1 \dots C_{\ell'}$

$A_S : \quad B_1 \wedge \dots \wedge B_\ell \supset C_1 \vee \dots \vee C_{\ell'} \vee A(a)$

$A_{S'} : \quad B_1 \wedge \dots \wedge B_\ell \supset C_1 \vee \dots \vee C_{\ell'} \vee \forall A(x)$

Theorem (LK Soundness)

Every sequent provable in LK is valid

Pf by induction on the number of sequents in proof.

Axiom $A \to A$ is valid

Induction step : use previous soundness lemma

<u>Soundness (Proof)</u>: By induction on the number of sequents in proof

Example: $\exists$ Left

Assume: $A(b), \Gamma \Rightarrow \Delta$ has an LK proof and is valid

show: $\exists x\, A(x), \Gamma \Rightarrow \Delta$ also Valid

By defn $\overline{A(b)} \vee \bar{\Gamma}_1 \vee \ldots \vee \bar{\Gamma}_k \vee \Delta_1 \vee \ldots \vee \Delta_k$ is valid

Let $\mathfrak{M}$ be any structure, σ any object assignment.

Show: $\mathfrak{M} \models \neg \exists x\, A(x) \vee \bar{\Gamma}_1 \vee \ldots \vee \bar{\Gamma}_k \vee \Delta_1 \vee \ldots \vee \Delta_k\, [\sigma]$ (*)

<u>case 1</u> $\mathfrak{M} \models \bar{\Gamma}_1 \vee \ldots \vee \bar{\Gamma}_k \vee \Delta_1 \vee \ldots \vee \Delta_k\, [\sigma]$. Then (*) holds

<u>Case 2</u> Case 1 does not hold.

Soundness (Proof): By induction on the number of sequents in proof

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Case 1 $\mathfrak{M} \models \overline{\Gamma}_1 \vee \ldots \vee \overline{\Gamma}_k \vee \Delta_1 \vee \ldots \vee \Delta_k [\sigma]$. Then (*) holds

Case 2 Case 1 does not hold.

Since b does not occur in Γ or Δ,

$$\mathfrak{M} \not\models \overline{\Gamma}_1 \vee \ldots \vee \overline{\Gamma}_k \vee \Delta_1 \vee \ldots \vee \Delta_k [\sigma(m/b)] \text{ for all } m \in M$$

Since $A(b), \Gamma \Rightarrow \Delta$ is valid, $\mathfrak{M} \models \overline{A(b)} [\sigma[m/b]] \; \forall m \in M$

Thus $M \models \neg \exists x A(x) [\sigma]$, & thus $\exists x A(x), \Gamma \Rightarrow \Delta$ is valid.