Announcements

- Office hours start next week:

  Monday 4-5 (Toni)

  Wednesday 1:30 - 2:30 (Oliver)

  LOCATION TBA (see courseworks or course webpage)

- Assignment 1 will be posted tonight (course webpage)

  due in 2 weeks (submit via gradescope)

Today

- Another proof system for propositional logic: PK
  - soundness of PK
  - completeness of PK
- Propositional compactness Theorem
- Derivational soundness/completeness of PK

Pages 9 – 17 of Lecture Notes

Gentzen's PK proof system

Lines in a PK proof are sequents

$$\underbrace{A_1, \ldots, A_k}_{\text{antecedent}} \rightarrow \underbrace{B_1, \ldots, B_r}_{\text{succedent}}$$

$A_1, \ldots, A_k, B_1, \ldots, B_r$ are propositional formulas

$\rightarrow$ is a new symbol (NOT part of language of propositional logic)

Gentzen's PK proof system

Lines in a PK proof are sequents

$$A_1, \ldots, A_k \rightarrow B_1, \ldots, B_r$$

Semantics:

$$A_1 \wedge A_2 \wedge \ldots \wedge A_k \supset B_1 \vee \ldots \vee B_r$$

the conjunction of the A_i's implies the disjunction of the B_i's

PK Rules

Structural Rules (cedents are sets)

Logical Rules (define the boolean connectives $\wedge, \vee, \neg$)

Cut Rule

STRUCTURAL RULES

	Left	Right
Weakening	$\dfrac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}$	$\dfrac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}$
Exchange	$\dfrac{\Gamma_1, A, B, \Gamma_2 \rightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \rightarrow \Delta}$	$\dfrac{\Gamma \rightarrow \Delta_1, A, B, \Delta_2}{\Gamma \rightarrow \Delta_1, B, A, \Delta_2}$
Contraction	$\dfrac{\Gamma, A, A \rightarrow \Delta}{\Gamma, A \rightarrow \Delta}$	$\dfrac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A}$

LOGICAL RULES

	Left	Right
$\neg$ Intro	$\dfrac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$	$\dfrac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$
$\wedge$ Intro	$\dfrac{A, B, \Gamma \Rightarrow \Delta}{(A \wedge B), \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, (A \wedge B)}$
$\vee$ Intro	$\dfrac{A, \Gamma \Rightarrow \Delta, \quad B, \Gamma \Rightarrow \Delta}{(A \vee B), \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, (A \vee B)}$

CUT RULE

$$\frac{\Gamma \to \Delta, A \qquad \Gamma, A \to \Delta}{\Gamma \to \Delta}$$

Axiom

$$A \to A$$

Example: A PK proof of a formula A is a PK proof of $\rightarrow A$

$$\dfrac{P \rightarrow P}{} \qquad \dfrac{Q \rightarrow Q}{} \qquad \text{weakening}$$

$$\dfrac{P \rightarrow Q, P}{P, \neg Q \rightarrow P} \qquad \dfrac{Q \rightarrow P, Q}{Q, \neg P \rightarrow Q} \qquad \neg \text{ Left}$$

$$P, \neg P, \neg Q \rightarrow \qquad Q, \neg P, \neg Q \rightarrow \qquad \neg\text{-Left}$$

$$(P \vee Q), \neg P, \neg Q \rightarrow \qquad \text{OR LEFT}$$

$$(P \vee Q), \neg P \wedge \neg Q \rightarrow \qquad \text{AND-Left}$$

$$(\neg P \wedge \neg Q) \rightarrow \neg (P \vee Q) \qquad \neg \text{ Right}$$

PK SOUNDNESS : Every sequent provable in PK is VALID

As in the propositional case, we first verify the soundness of all rules + then prove PK soundness by induction

Lemma (Soundness of Rules)
For every rule of PK, if all top sequents are valid, then the bottom sequent is valid
also the axiom is valid

PK soundness If S has a PK proof, then A_S is valid

PK COMPLETENESS: Every valid propositional sequent has a PK proof

Main idea: again we will give an algorithm that will produce a PK proof for an valid sequent

PK Algorithm (on input $\Gamma \to \Delta$):

0.) Write sequent at bottom (of proof tree)

1.) Repeatedly: pick an outermost connective in a formula in a leaf sequent of current proof & apply the rule for that connective (in reverse)

2.) Continue until all leaf sequents consist of just atoms

Show: If we run algorithm on a valid sequent $\Gamma \to \Delta$, then at end, all leaf sequents must contain an atom occuring both on left + right — ie $A, B, C \to A, D$

Then can finish proof by applying weakening (in reverse)

ie.
$$\frac{A \to A}{A, B, C \to A, D}$$

PK completeness (cont'd)

Key Property is the INVERSION PRINCIPLE:

each PK rule except weakening has the property that $\forall$ truth assignments τ, if τ satisfies bottom sequent, then τ satisfies both upper sequents

* called inversion since it is the reverse direction of what we needed to prove soundness: $\forall\tau$ if τ satisfies both upper sequents, then τ satisfies lower sequent

PK completeness

- If $\Gamma \to \Delta$ is valid, by Inversion Property, all leaf sequents generated in step 1 of Algorithm are VALID, and have one less connective than sequent below

- Thus eventually step 1 halts, where each leaf sequent involves only atoms and each leaf sequent is valid

  $\therefore$ each leaf sequent looks like $A, \Gamma \to A, \Delta$

  ie has an atom A on both Left + Rt sides

PK completeness

Claim Let π be the output of PK algorithm on input $\Gamma \to \Delta$.
If $\Gamma \to \Delta$ is valid then every leaf sequent of π contains only atomic formulas (propositional variables) on left/right of "$\to$" and furthermore there exists some atomic formula occurring on both the left & right

Pf Suppose for sake of contradiction that some leaf-sequent of π is: $x_1, \ldots, x_k \to y_1, \ldots, y_\ell$
where $\{x_1, \ldots, x_k\} \cap \{y_1, \ldots y_\ell\} = \phi$. Then the truth assignment τ that sets $x_1 = x_2 = \ldots = x_k = 1$, $y_1 = y_2 = \ldots = y_\ell = 0$ falsifies the sequent which contradicts fact that $\Gamma \to \Delta$ valid implies all leaf sequents of π are also valid.

Cut-Elimination Theorem for PK

If $\Gamma \to \Delta$ has a PK proof, then it has a proof with no use of the cut rule.

Derivational Soundness + Completeness of PK

Definition Let Φ be a set of sequents, S a sequent

A PK-Φ proof of S is a PK-proof of S from Φ and axioms of PK.

Theorem. Let Φ be a set of (possibly infinite) sequents. Then $\Phi \vdash S$ iff S has a (finite) PK-Φ proof

Propositional Compactness

Theorem (Form 2, see Notes for 2 other equivalent forms)

Let Φ be a set of (possibly infinite) formulas

$\Phi \models A$ iff A is a logical consequence of a finite subset of Φ

We'll assume this for now and prove it after Proof of 3 equivalent forms of Compactness as homework

Proof (Derivational Soundness/completeness)

By compactness, it suffices to prove the case where Φ is finite

- Let $\Phi = \{S_1, .., S_k\}$, and suppose $\Gamma \Rightarrow \Delta$ is a logical consequence of $\{S_1, .., S_k\}$. Thus

$$(*) \quad \Gamma, A_{S_1}, .., A_{S_k} \Rightarrow \Delta \quad \text{is valid}$$

- Thus by PK completeness, $(*)$ has a PK proof
- Derive $\Gamma \Rightarrow \Delta$ from $(*)$ and $\Rightarrow A_{S_1}, .., A_{S_k}$

Derive $\Gamma \to \Delta$ from $\{ \to A_{s_1}, \to A_{s_2}, \to A_{s_3}, (*) \}$

PK proof of

$$\Gamma, A_{s_1}, A_{s_2}, A_{s_3} \to \Delta$$

$$\frac{\to A_{s_1}}{\Gamma, A_{s_2}, A_{s_3} \to \Delta, A_{s_1}} \text{(weakening)}$$

$$\Gamma, A_{s_2}, A_{s_3} \to \Delta \quad \text{(cut)}$$

$$\frac{\to A_{s_2}}{\Gamma, A_{s_3} \to \Delta, A_{s_2}} \text{(weakening)}$$

$$\Gamma, A_{s_3} \to \Delta \quad \text{(cut)}$$

$$\frac{\to A_{s_3}}{\Gamma \to A_{s_3}, \Delta} \text{(weakening)}$$

$\Gamma \to \Delta$

$$\Gamma \to \Delta \quad \text{(cut)}$$

Proof (Propositional Compactness)

Suppose $\Phi \vDash A$. Then $\overbrace{\Phi, \neg A}^{\Psi}$ is unsatisfiable

<u>show</u>: If Ψ is UNSAT, then some finite subset of Ψ is UNSAT (Form 1)

<u>Pf sketch</u> Assume the set of underlying atoms in Ψ is countable: $P_1, P_2, \ldots$

- Make decision tree T that queries P_1 at layer 1, then P_2 at layer 2, etc.

- Each path in T corresponds to a complete truth assignment

- Prune T to T':

  For every node v of T, remove subtree rooted below v if partial truth assignment from root to v falsifies some formula $f \in \varphi$. Label v by f

- Every path in T' is finite (since φ unsat, so $\forall$truth ass to all vars, some $f \in \varphi$ is falsified, and each $f \in \varphi$ is finite)

- By König's Lemma, T' is finite

<u>König's Lemma</u> If T' is a rooted binary tree, where every branch/path of T' is finite, Then T' is finite.

- Thus, the formulas $\Psi' \subseteq \Psi$ labelling the leaves of T' form a finite subset of Ψ, and thus Ψ' is UNSAT & finite subset of Ψ.