## Announcements

- Office hours start Next Week:
   Monday 4-S (Toni)
   Wednesday 1:30 Z:30 (Oliver)
   LOCATION TBA (see courseworks or course webpage)
  - Assignment 1 will be posted to night (course webpage)
     due in 2 weeks (submit via gradescope)

Today

- Another proof system for propositional logic: PK soundness of PK completeness of PK
  - · Propositional compactness Theorem
  - · Derivational soundness/completeness of PK

Pages 9-17 of Lecture Notes

gentzen's PK proof system

Lines in a PK proof are sequents  

$$A_{1,...,}A_{k} \rightarrow B_{1,...,}B_{r}$$
  
antecedent succedent

gentzen's PK proof system

Lines in a PK proof are sequents  

$$A_{1,...,}A_{k} \longrightarrow B_{1,...,}B_{r}$$

Semantics:  

$$A_1 \land A_2 \land \dots \land A_k \supseteq B_1 \lor \dots \lor B_r$$
  
The conjunction of the  $A_2$ 's implies  
the disjunction of the  $B_1$ 's

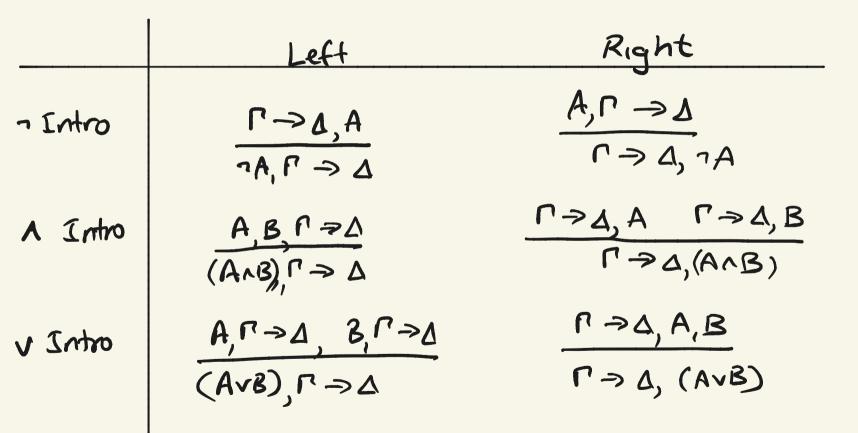
PK Rules

Cut Rule

STRUCTURAL RULES

	Left	Right
Weakening	$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$	$\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}$
Exchange	$\frac{\Gamma_{,,A,B,\Gamma_{2}} \rightarrow \Delta}{\Gamma_{,,B,A,\Gamma_{2}} \rightarrow \Delta}$	$\frac{\Gamma \rightarrow \Delta_{1}, A, B, \Delta_{2}}{\Gamma \rightarrow \Delta_{1}, B, A, \Delta_{2}}$
Contraction	$\frac{\Gamma, A, A \rightarrow \Delta}{\Gamma, A \rightarrow \Delta}$	$\frac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A}$

LOGICAL RULES



 $\frac{CUTRULE}{\Pi \rightarrow \Delta, A \quad \Pi, A \rightarrow \Delta}{\Pi \rightarrow \Delta}$ 



A⇒A

Example: A PK a PK proof of -		formula A is
P>P	$Q \rightarrow Q$	wea kening
P->Q,P	$Q \rightarrow P_{Q}$	7 left -
$\frac{P, nQ \rightarrow P}{P}$	$Q, P \rightarrow Q$	Left
$P, \neg P, \neg Q \rightarrow $	Q,79,7Q.	> OR LEFT
$\frac{(P \sim Q)}{(P \sim Q)}, \neg P, \neg$		AND - Left
$\frac{(PrQ)}{r}$		7 Right
(7P~ ~Q) -	$ \rightarrow \neg (r \lor q) $	

PK SOUNDNESS : Every sequent provable in PK is VALID

As in the propositional case, we first verify the soundness of all rules + then prove PK soundness by induction

Lemma (soundness of Rules) For every rule of PK, if all top sequents are valid, then the bottom sequent-is valid also the axiom is valid PK soundness If S has a PK prob, men As is valid PK COMPLETENESS: Every valid propositionial sequent has a PK proof

Main idea: again we will give an algorithm that will produce a PK proof for an valid sequent PK Algorithm (on input  $\Gamma \rightarrow \delta$ ): 0.) Write sequent at bottom (of proof tree) 1.) Repeatedly: pick an outermost connective in a formula in a reaf sequent of current proof + apply the rule for that convective (in reverse) 2.) contrinue until all leaf sequents consist of just atoms

Show: If we run algorithm on a valid sequent  $\Gamma \rightarrow \Delta$ , then at end, all leaf sequents must contain an atom occurring both on left + right — ie A,B,C  $\rightarrow$  A,D

Then can finish proof by applying weakening (in reverse) (e.  $A \rightarrow A$ A = A $A, B, C \rightarrow A, D$  PK completeness (cont'd)

Key Property is the INVERSION PRINCIPLE: each PK rule except weakening has the property that V truth assignments  $\mathcal{P}_{if}$  if  $\mathcal{T}$  satisfies bottom sequent, then  $\mathcal{T}$  satisfies both upper sequents

t called inversion since it is the reverse direction of what we needed to prove soundness: IT if Tratisfies both upper sequents, then T satisfies Cover sequent

PK completeness

## • If 1-70 is valid, by Inversion Property, all leaf sequents generated in step (1) of Algorithm are VALID and have one less connective than sequent below o Thus eventually step (1) halts, where each leaf sequent involves only atoms and each leaf sequent is valid : each leaf sequent looks like D,A ← J,A ie has an atom A on both Left + Rt sides

PK completeness

claim Let T be the output of PK algorithm on input ~>1. If  $T \rightarrow \Delta$  is valid then every leaf sequent of TT contains only atomic formulas (propositional variables) on Left/right of ">" and furthermore, there exists some atomic formula occurring on both the left + right PE suppose for sake of contradiction that some leasequent of TT is: X1, ..., XK > Y1, ..., Ye where {x, , , x, } n {y, - ye} = \$, Then the truth assignment T' that sets x=x====Xk=1, Yi=Yz===Ye=0 falsifies the sequent which contradicts buck that T=0 Valid implies all leap sequents of TT are also valid.

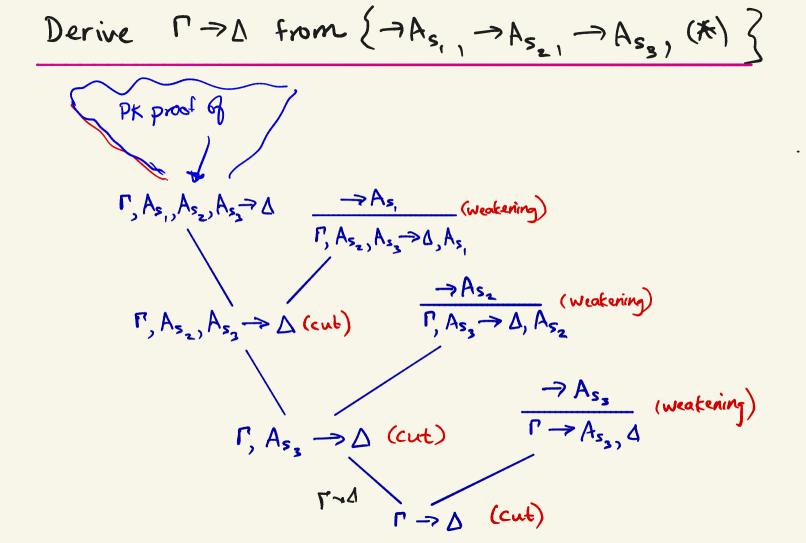
Derivational Soundness + Completeness & PK

Theorem. Let 
$$\overline{\Phi}$$
 be a set of (possibly infinite)  
sequents. Then  $\overline{\Phi} \vdash S$  iff  
s has a (finite)  $PK-\overline{\Phi}$  proof

Propositional Compactness

Q We'll assume this for Now and prove it after Proof of 3 equivalent forms of compactness as homework Proof (Derivational Soundness/ completeness)

Thus by PK completeness, (\*) has a PK proof Derive  $\Gamma = A$  from (\*) and  $= A_{s_1}$ , ...,  $A_{s_k}$ 



Proof (Propositional compactness) Suppose  $\overline{\Phi} \neq A$ . Then  $\overline{\Phi}$ ,  $\overline{A}$  is unsatisfiable show: If Y's UNSAT, then some finite subset of Y's UNSAT (Form 1) Pf sketch Assume the set of underlying atoms in V is countable: P1, P2, .... • Make decision tree That queries p, at layer 1, then P2 at layer z, etc.

 Each path in T corresponds to a complete touth assignment

For every node v of T, remove subtree rooted below v if partial truth assignment from root to v falsifies some formula fe P. Label v by f

• Every path in T' is finite (since  $\psi$  unsat, so  $\forall$ truth ass to all vars, some  $f \in \psi$  is falsified, and each  $f \in \psi$  is finite)

- By König's Lemma, T' is finite

König's Lemma If T'is a rooted  
binary tree, where every branch/path  
of T is finite, Then T'is finite.  
Thus, the formulas 
$$\Psi' \in \Psi$$
 labelling the leaves  
of T' form a finite subset of  $\Psi$ ,  
and thus  $\Psi'$  is WSAT + finite  
subset of  $\Psi$ .