Last class:

1. Intro
2. Propositional logic

   syntax / semantics

   Resolution: proof system for propositional logic

   - soundness
   - completeness

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   Resolution: proof system for propositional logic

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Pages 1-9 of Lecture Notes, plus supplementary notes on Resolution

Announcements

- Office hours start next week:

  Monday 4-5 (Toni)

  Wednesday 1:30 - 2:30 (Oliver)

  Likely by zoom

- Assignment 1 will be posted this week!

Today

- Another proof system for propositional logic: PK
  - soundness of PK
  - completeness of PK
- Propositional compactness Theorem
- Derivational soundness/completeness of PK

Pages 9 – 17 of Lecture Notes

THE (PROPOSITIONAL) SEQUENT CALCULUS

- A second sound + complete system for propositional logic

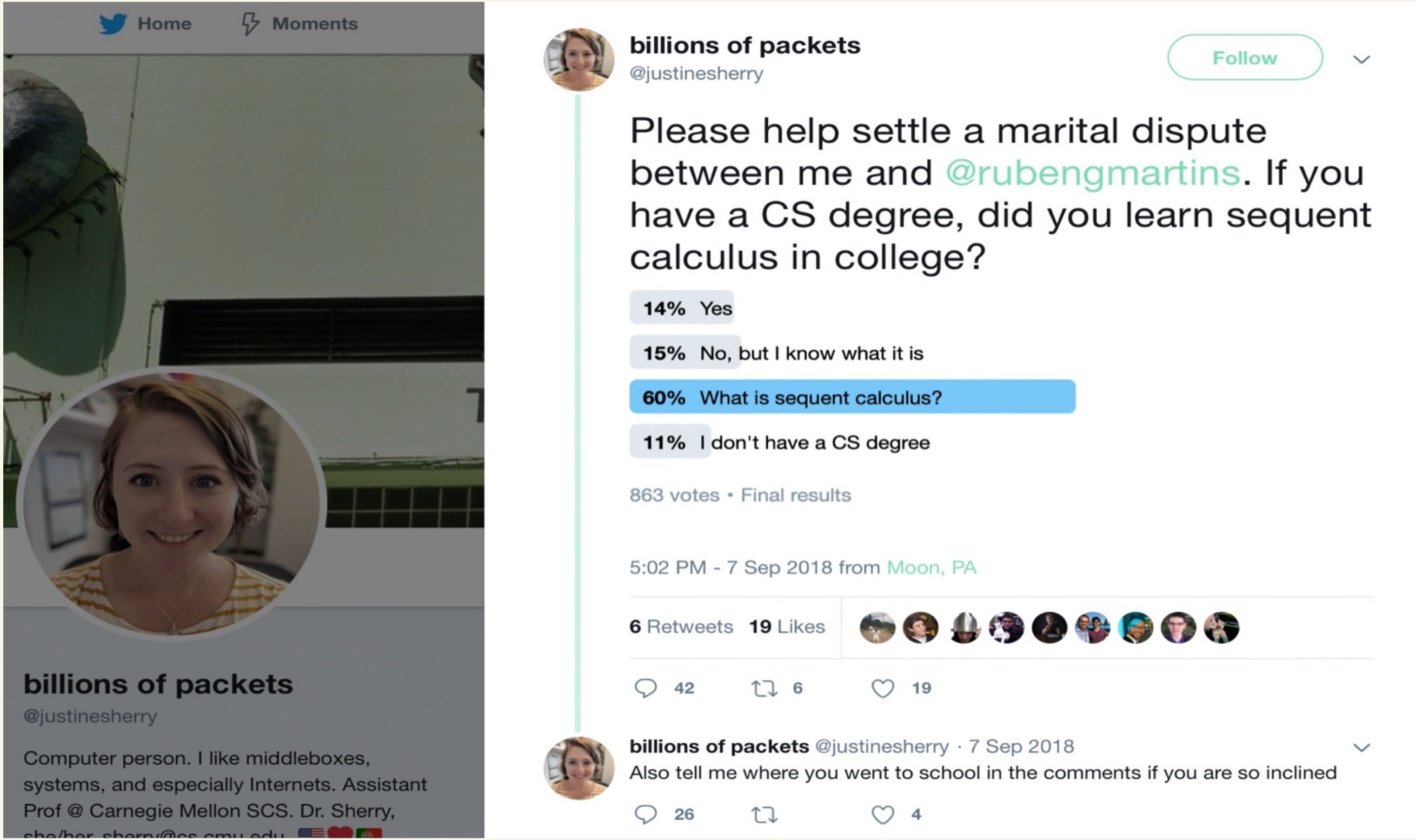
- More natural for proving theorems (but more difficult as basis for automated theorem proving / SAT solving)

Why I love the Sequent Calculus

- Rules are very natural – 2 for each boolean connective
- Cut Rule (Modus Ponens) Not needed for completeness. (This formulation requires meta-symbol "$\rightarrow$")

  This greatly simplifies/clarifies completeness proofs for propositional and first order logic

Sequent Calculus goes viral on Twitter



Home Moments

ions of packets

tinesherry

puter person. I like middleboxes,
ems, and especially Internets. Assistant
@ Carnegie Mellon SCS. Dr. Sherry,

4 17

Pierce Darragh @_pdarragh · 7 Sep 2018

Pretty sure it's the calculus of how to put shiny plastic things on clothing

1 8

John Regehr @johnregehr · 7 Sep 2018

YES!!!

2

kat 猫 ✨🐙✨ @wirehead2501 · 7 Sep 2018

Replying to @justinesherry @rubengmartins

@chrisamaphone no; i have a CS BS and comp eng MS and only had to take Calc I and II. don't think i've ever heard the term "sequent"

2

Chris Martens @chrisamaphone · 8 Sep 2018

But surely you have heard its plural, as in, “a sequents of events” 😬

1 6

jason reed @jcreed · 8 Sep 2018

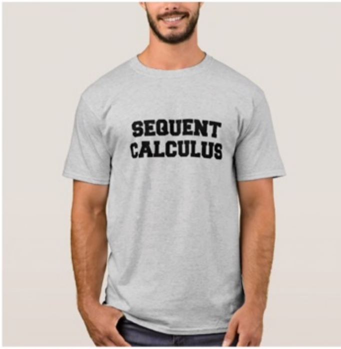
if you are against bedazzling: sequin't

1 3

jason reed @jcreed · 8 Sep 2018

oh wait I have confused sequins with rhinestones, my bad

3



Sequent Calculus Standard Shirt

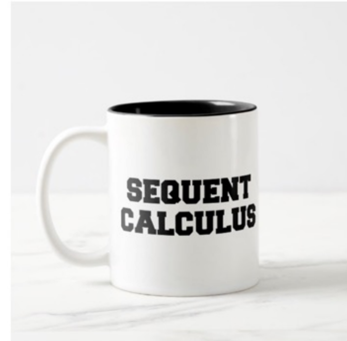
$18.95

15% Off with code GANGSALLHERE

Sequent Calculus Fitted Shirt

$17.95

15% Off with code GANGSALLHERE



Sequent Calculus Mug

$16.95

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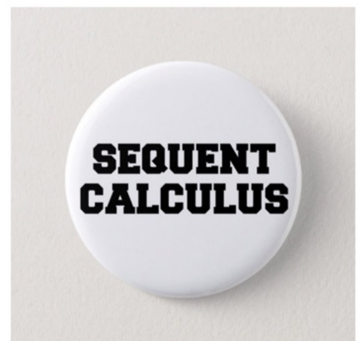
Sequent Calculus Tote Bag

$9.95

15% Off with code GANGSALLHERE

German Fitted Sequent Calculus Shirt

German Sequent Calculus Shirt



Sequent Calculus Pin

Gentzen's PK proof system

Lines in a PK proof are sequents

$$\underbrace{A_1, \ldots, A_k}_{\text{antecedent}} \rightarrow \underbrace{B_1, \ldots, B_r}_{\text{succedent}}$$

$A_1, \ldots, A_k, B_1, \ldots, B_r$ are propositional formulas

$\rightarrow$ is a new symbol (NOT part of language of propositional logic)

Gentzen's PK proof system

Lines in a PK proof are **sequents**

$$A_1, \ldots, A_k \rightarrow B_1, \ldots, B_r$$

Semantics:

$$A_1 \wedge A_2 \wedge \ldots \wedge A_k \supset B_1 \vee \ldots \vee B_r$$

the conjunction of the A_i's implies the disjunction of the B_i's

Gentzen's PK proof system

Lines in a PK proof are **sequents**

$$A_1, \ldots, A_k \rightarrow B_1, \ldots, B_r \qquad \stackrel{d}{=} S$$

<u>Semantics</u> :

$$A_1 \wedge A_2 \wedge \ldots \wedge A_k \supset B_1 \vee \ldots \vee B_r \qquad \stackrel{d}{=} A_S$$

the conjunction of the A_i's implies the disjunction of the B_i's

$$S \stackrel{d}{=} A_1, \ldots, A_k \to B_1, \ldots, B_j$$

$$A_S : \neg(A_1 \wedge \ldots \wedge A_k) \vee (B_1 \vee \ldots \vee B_j)$$

Gentzen's PK proof system

Lines in a PK proof are **sequents**

$S \stackrel{d}{=} A_1, \ldots, A_k \rightarrow B_1, \ldots, B_q$

$A_S \doteq \neg(A_1 \wedge \ldots \wedge A_k)$

0 or False

1 or True

Convention: Empty conjunction (antecedent empty) $\rightarrow$ 1

Empty disjunction (succedent empty) $\rightarrow$ 0

PK Rules

Intuitively: Structural Rules (cedents are sets)

Logical Rules (define the boolean connectives $\land, \lor, \neg$)

Cut Rule

STRUCTURAL RULES

	Left	Right
Weakening	$\dfrac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$
Exchange	$\dfrac{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta_1, A, B, \Delta_2}{\Gamma \Rightarrow \Delta_1, B, A, \Delta_2}$
Contraction	$\dfrac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$

LOGICAL RULES

	Left	Right
¬ Intro	$\dfrac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$	$\dfrac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$
∧ Intro	$\dfrac{A, B, \Gamma \Rightarrow \Delta}{(A \wedge B), \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, (A \wedge B)}$
∨ Intro	$\dfrac{A, \Gamma \Rightarrow \Delta, \quad B, \Gamma \Rightarrow \Delta}{(A \vee B), \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, (A \vee B)}$

So we can think of

$$A_1, \ldots, A_k \rightarrow B_1, \ldots, B_j$$

as $\rightarrow \neg A_1 \vee \neg A_2 \vee \neg A_3 \vee \ldots \vee \neg A_k \vee B_1 \vee \ldots \vee B_j$

CUT RULE

$$\frac{\Gamma \to \Delta, A \qquad \Gamma, A \to \Delta}{\Gamma \to \Delta}$$

Axiom

$$A \to A \quad \equiv \quad \underbrace{\neg A \vee A}_{\text{law of excluded middle}}$$

Example: A PK proof of a formula A is a PK proof of $\rightarrow A$

$$\dfrac{P \rightarrow P}{P \rightarrow Q, P} \qquad \dfrac{Q \rightarrow Q}{Q \rightarrow P, Q} \qquad \text{weakening}$$

$$\dfrac{P \rightarrow Q, P}{P, \neg Q \rightarrow P} \qquad \dfrac{Q \rightarrow P, Q}{Q, \neg P \rightarrow Q} \qquad \neg \text{ Left}$$

$$\dfrac{P, \neg Q \rightarrow P}{P, \neg P, \neg Q \rightarrow} \qquad \dfrac{Q, \neg P \rightarrow Q}{Q, \neg P, \neg Q \rightarrow} \qquad \neg\text{-Left}$$

$$\dfrac{P, \neg P, \neg Q \rightarrow \qquad Q, \neg P, \neg Q \rightarrow}{(P \vee Q), \neg P, \neg Q \rightarrow} \qquad \text{OR LEFT}$$

$$\dfrac{(P \vee Q), \neg P, \neg Q \rightarrow}{(P \vee Q), \neg P \wedge \neg Q \rightarrow} \qquad \text{AND-Left}$$

$$\dfrac{(P \vee Q), \neg P \wedge \neg Q \rightarrow}{(\neg P \wedge \neg Q) \rightarrow \neg (P \vee Q)} \qquad \neg \text{ Right}$$

PK SOUNDNESS : Every sequent provable in PK is VALID

As in the propositional case, we first verify the soundness of all rules + then prove PK soundness by induction

Lemma (Soundness of Rules)
For every rule of PK, if all top sequents are valid, then the bottom sequent is valid
also the axiom is valid

PK soundness If S has a PK proof, then A_S is valid

Example

$$\frac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, (A \wedge B)}$$ AND-rt rule

Let $\Gamma = C_1, \ldots, C_k$
$\Delta = D_1, \ldots, D_\ell$

Show:

Let τ be any truth ass to atoms in $\Gamma, \Delta, (A \vee B)$

If τ satisfies $\neg(C_1 \wedge \ldots \wedge C_k) \vee (D_1 \vee \ldots \vee D_\ell) \vee A$] formula corresponding to $\Gamma \to \Delta, A$

and τ satisfies $\neg(C_1 \wedge \ldots \wedge C_k) \vee (D_1 \vee \ldots \vee D_\ell) \vee B$] formula for $\Gamma \to \Delta, B$

Then τ satisfies $\neg(C_1 \wedge \ldots \wedge C_k) \vee (D_1 \vee \ldots \vee D_\ell) \vee (A \wedge B)$

PK COMPLETENESS: Every valid propositional sequent has a PK proof

Main idea: again we will give an algorithm that will produce a PK proof for any valid sequent

Algorithm: Write sequent at bottom (of proof)

1. Repeatedly: pick an outermost connective in a formula in a leaf sequent of current proof & apply the rule for that connective (in reverse)
2. Continue until all leaf sequents consist of just atoms

<u>Show</u>: If we run algorithm on a <u>valid</u> sequent $\Gamma \to \Delta$, then at end, all leaf sequents must contain an atom occuring both on left + right — ie $A, B, C \to A, D$

Then can finish proof by applying weakening (in reverse)

ie. $$\frac{A \to A}{A, B, C \to A, D}$$

PK completeness (cont'd)

Key Property is the **INVERSION PRINCIPLE**:

each PK rule except weakening has the property that $\forall$ truth assignments τ, if τ satisfies bottom sequent, then τ satisfies both upper sequents

* called inversion since it is the reverse direction of what we needed to prove soundness: $\forall \tau$ if τ satisfies both upper sequents, then τ satisfies lower sequent

PK completeness

- If $\Gamma \to \Delta$ is valid, by Inversion Property, all leaf sequents generated in step 1 of Algorithm are VALID, and have one less connective than sequent below

- Thus eventually step 1 halts, where each leaf sequent involves only atoms and each leaf sequent is valid

  $\therefore$ each leaf sequent looks like $A, \Gamma \to A, \Delta$

  ie has an atom A on both Left & Rt sides

PK completeness

If Not: Say $A, B \to C, D, E$ is a leaf. Then this sequent is Not valid!

Cut-Elimination Theorem for PK

If $\Gamma \rightarrow \Delta$ has a PK proof, then it has a proof with no use of the cut rule.