Last class:

1. Intro
2. Propositional logic

    Syntax / semantics

    Resolution: proof system for propositional logic

    - soundness
    - completeness

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Pages 1-9 of Lecture Notes, plus supplementary notes on Resolution

Today

- Finish Resolution

- Another proof system for propositional logic: PK
  - soundness of PK
  - completeness of PK

- Propositional compactness Theorem

- Derivational soundness/completeness of PK

Pages 9–17 of Lecture Notes

Definitions (From last class)

τ satisfies A iff $A^\tau = T$

τ satisfies a set Φ of formulas iff

τ satisfies A for all $A \in \Phi$

Φ is satisfiable iff $\exists \tau$ that satisfies Φ

otherwise Φ is unsatisfiable

$\Phi \models A$ (A is a logical consequence of Φ) iff

$\forall \tau$ [τ satisfies $\Phi \Rightarrow \tau$ satisfies A]

$\models A$ (A is valid or A is a tautology) iff

$\forall \tau$ [τ satisfies A]

Resolution: Proof System for Prop Logic

- Resolution is basis for most automated theorem provers
- Proves that formulas are unsatisfiable (recall F is a tautology iff $\neg F$ is valid)
- Formulas have to be in a special form: CNF

$$\underbrace{(x_1 \vee x_2 \vee \bar{x}_3)}_{C_1} \wedge \underbrace{(\bar{x}_2 \vee x_4)}_{C_2} \wedge \underbrace{(\bar{x}_4)}_{C_3} \wedge \underbrace{(x_1 \vee x_3)}_{C_4} \wedge \underbrace{(x_1)}_{C_5}$$

Converting a formula to CNF

- obvious method (deMorgan) could result in an exponential blowup in size

  Example $(x_1 \wedge x_2) \vee (x_3 \wedge x_4) \vee (x_5 \wedge x_6) \vee \dots (\quad)$

- Better method : SAT THEOREM

  There is an efficient method to transform any propositional formula F into a CNF formula g such that F is satisfiable iff g is satisfiable

SAT THEOREM : proof by example

$$F : \underbrace{\underbrace{(Q \wedge R)}_{P_B} \vee \neg Q}_{P_A}$$

new variables

$$g : (P_B \Leftrightarrow (Q \wedge R)) \wedge (P_A \Leftrightarrow P_B \vee \neg Q) \wedge (P_A)$$

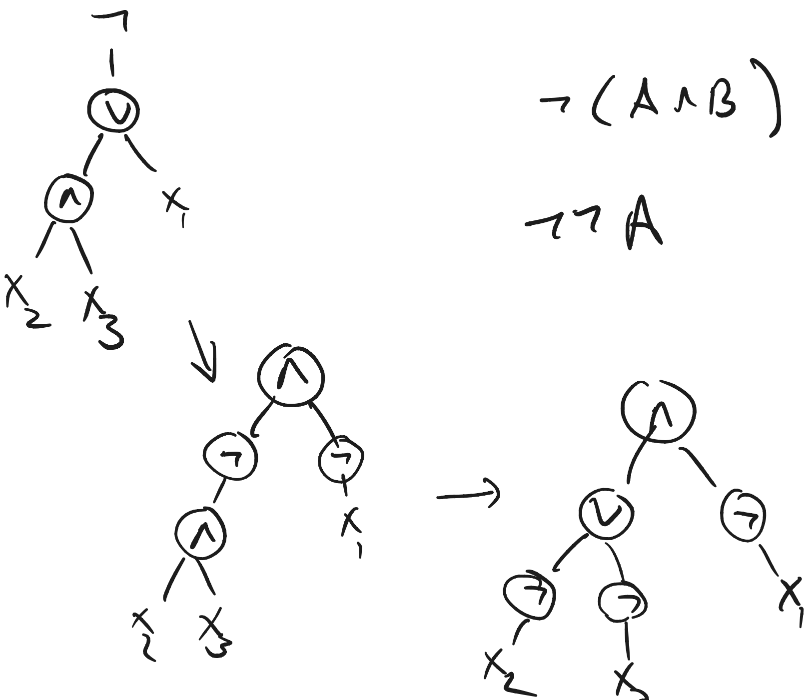
$$(\neg P_B \vee Q)(\neg P_B \vee R)(\neg Q \vee \neg R \vee P_B)$$

Demorgan's Rules to push negations to leaves:

$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$

$$\neg(A \wedge B) \equiv \bar{A} \vee \bar{B}$$

$$\neg\neg A \equiv A$$



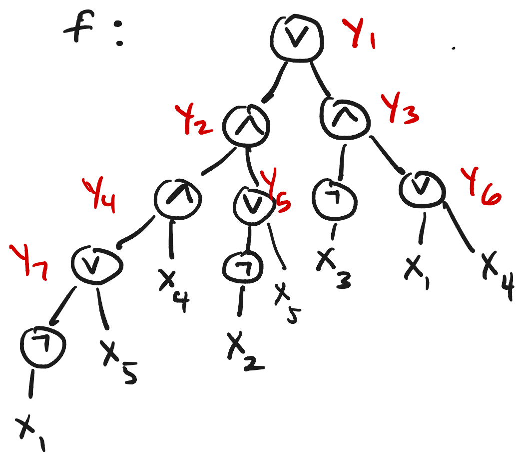
$$\overline{(A \vee B)} \wedge \overline{(C \vee D)} \wedge \overline{(E \vee F)}$$

$$\equiv (A \wedge C \wedge E) \vee (\quad)$$

Theorem Let f be a formula of size m (m leaves) with n variables $x_1 \dots x_n$. Then there exists an equivalent 3CNF formula g with $O(m)$ variables and size $O(m)$

Example

f :



$$
\begin{array}{lcl}
g: \; (y_1) \wedge & & (y_1) \\
(y_1 \leftrightarrow y_2 \vee y_3) \wedge & \Longleftrightarrow & (\neg y_1 \vee y_2 \vee y_3)(\neg y_2 \vee y_1)(\neg y_3 \vee y_1) \\
(y_2 \leftrightarrow y_4 \wedge y_5) \wedge & & (\neg y_2 \vee y_4)(\neg y_2 \vee y_5)(\neg y_4 \vee \neg y_5 \vee y_2) \\
(y_3 \leftrightarrow \neg x_3 \wedge y_6) \wedge & & \vdots \\
(y_4 \leftrightarrow y_7 \wedge x_4) \wedge & & \\
(y_5 \leftrightarrow \neg x_2 \vee x_5) \wedge & & \\
(y_6 \leftrightarrow x_1 \vee x_4) \wedge & & \vdots \\
(y_7 \leftrightarrow \neg x_1 \vee x_5) & & (\neg y_7 \vee \neg x_1 \vee x_5)(x_1 \vee y_7)(\neg x_5 \vee y_7)
\end{array}
$$

RESOLUTION

Start with CNF formula $F = C_1 \wedge C_2 \wedge \dots \wedge C_m$
view F as a set of clauses $\{C_1, C_2, \dots, C_m\}$

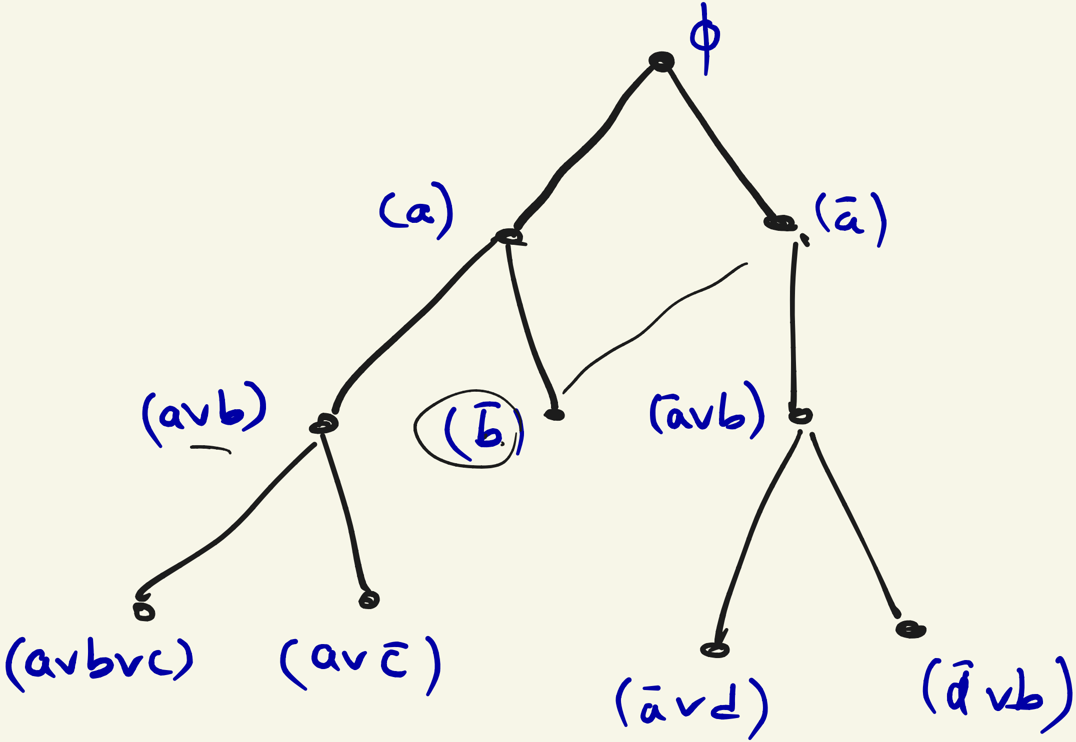
Resolution Rule :

$(A \vee x), (B \vee \bar{x})$ derive $(A \vee B)$

A Resolution Refutation of F is a sequence of clauses $D_1, D_2, \dots, D_q$ such that:
each D_i is either a clause from F, or follows from 2 previous clauses by Resolution rule,
and final clause $D_q = \phi$ (the empty clause)

Resolution Refutation

$F = (a \vee b \vee c)(a \vee \bar{c})(\bar{b})(\bar{a} \vee d)(\bar{d} \vee b)$



a clause C is derived from a set Φ of clauses if $\exists$ a sequence of clauses $D_1, \ldots, D_m$ s.t.

1. Each D_i is either $\in \Phi$ or follows from 2 previous $D_{i'}, D_{i''}$ $i', i'' < i$ clauses by the Res rule

2. D_m is C

If $C = \emptyset$ (the empty clause) is derivable from Φ then Φ has a Res refutation

Resolution Soundness

Fact: If C_1, C_2 derive C_3 by Resolution rule,

then $C_1, C_2 \models C_3$

From above Fact we can prove:

RESOLUTION SOUNDNESS THEOREM

If a CNF formula F has a RES refutation, then F is unsatisfiable

RESOLUTION COMPLETENESS THM

Every unsatisfiable CNF formula F has a RESOLUTION Refutation

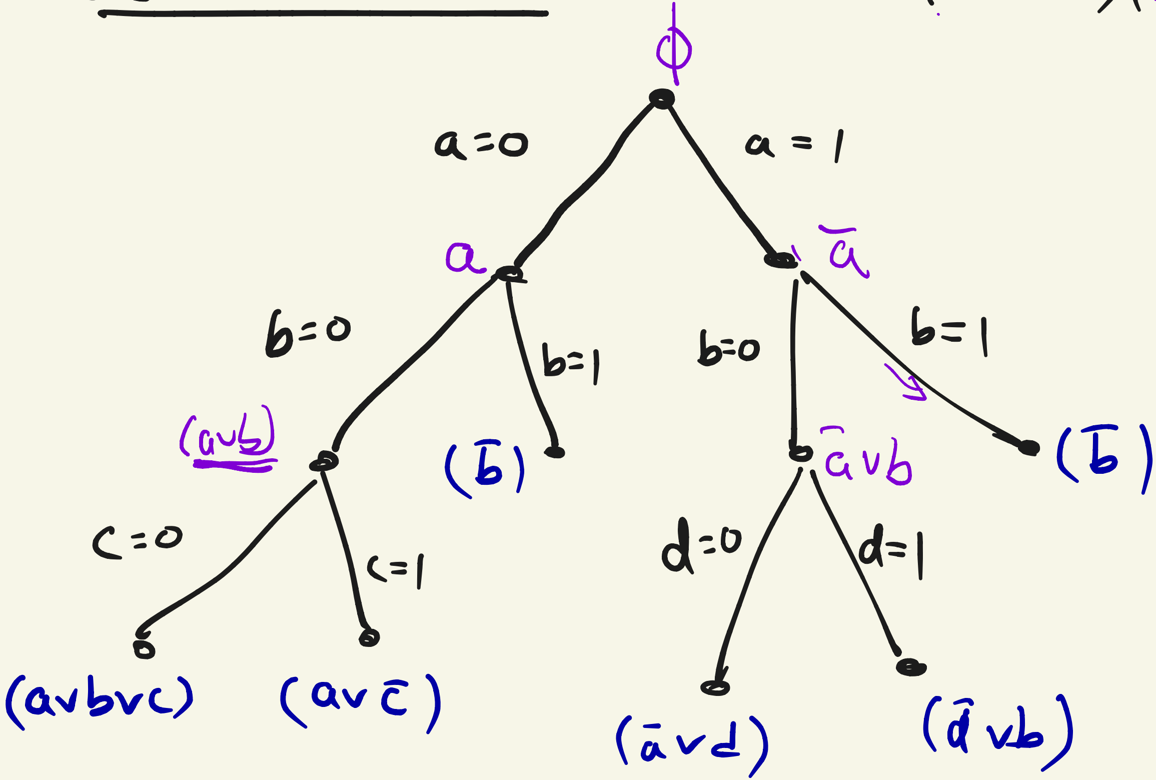
Proof idea

We describe a canonical procedure for obtaining a RES refutation for F

The procedure exhaustively tries all truth ass's – via a decision tree
then we show that any such decision tree can be viewed as a RES refutation.

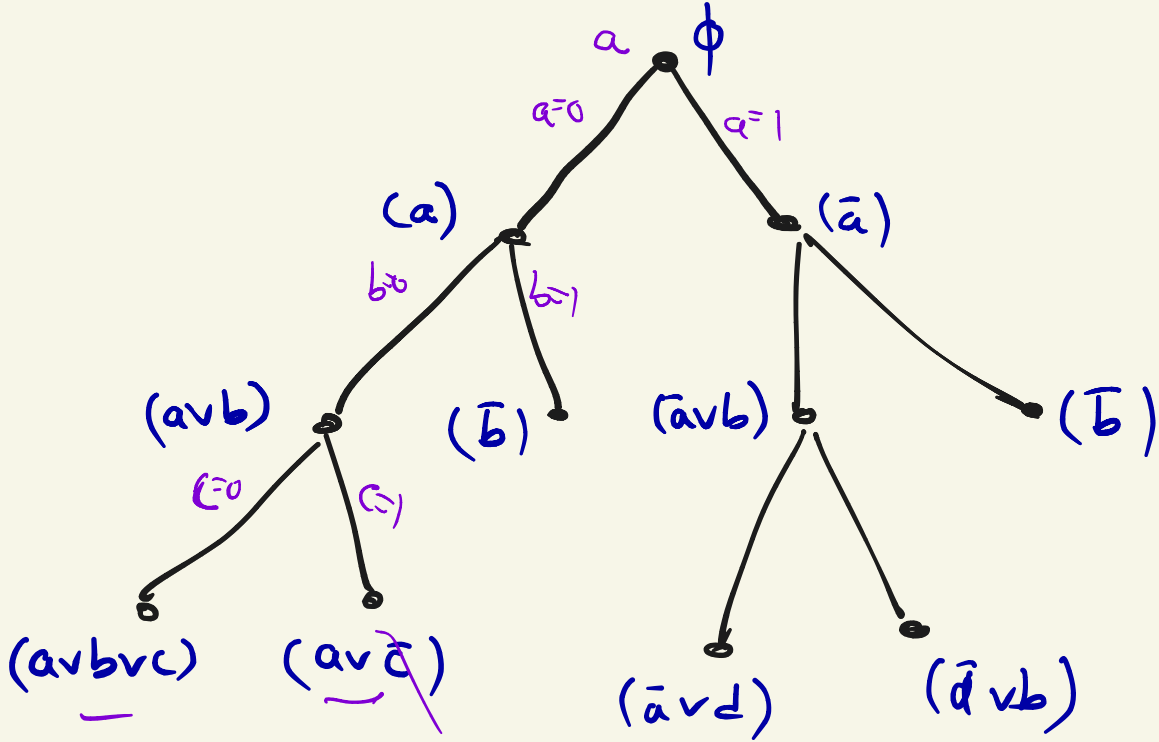
DECISION TREES

$F = (a \vee b \vee c)(a \vee \bar{c})(\bar{b})(\bar{a} \vee d)(\bar{d} \vee b)$



≡ Resolution Refutation

$F = (a \vee b \vee c)(a \vee \bar{c})(\bar{b})(\bar{a} \vee d)(\bar{d} \vee b)$



COMPLEXITY OF RESOLUTION REFUTATIONS

Let Π be a RES refutation of CNF F over $x_1 \dots x_n$

SIZE(Π) = Number of clauses in Π

Π is tree-like if directed acyclic graph (ignoring initial clauses of F) is a tree

Upper bound: $\text{size}(\Pi) \leq 2^n$ Why?

Lower bound: Are there unsat formulas $\{F_n\}_{n \geq 1}$ requiring exponential-sized RES proofs?