Review of Definitions

$\mathcal{L}_A = \{0, s, +, \cdot\,; =\}$ Language of arithmetic

$\Phi_0 =$ all $\mathcal{L}_A$- sentences

$TA = \{A \in \Phi_0 \mid \mathbb{N} \models A\}$ True Arithmetic

A theory Σ is a set of sentences (over $\mathcal{L}_A$) closed under logical consequence

- We can specify a theory by a subset of sentences that logically implies all sentences in Σ

Σ is consistent iff $\Phi_0 \neq \Sigma$ (iff $\forall A \in \Phi_0$, either A or $\neg A$ Not in Σ)

Σ is complete iff Σ is consistent and $\forall A$ either A or $\neg A$ is in Σ

Σ is sound iff $\Sigma \subseteq TA$

Let $\mathfrak{M}$ be a model/structure over $\mathcal{L}_A$

$Th(\mathfrak{M}) = \{ A \in \Phi_0 \mid \mathfrak{M} \models A \}$

$Th(\mathfrak{M})$ is complete (for all structures $\mathfrak{M}$)

Note $TA = Th(\mathbb{N})$ is complete, consistent, & sound

$VALID = \{ A \in \Phi_0 \mid \models A \}$ ⟵ smallest theory

Let Σ be a theory

Σ is <u>axiomatizable</u> if there exists a set $\Gamma \subseteq \Sigma$

such that ① Γ is recursive

② $\Sigma = \{ A \in \Phi_0 \mid \Gamma \models A \}$

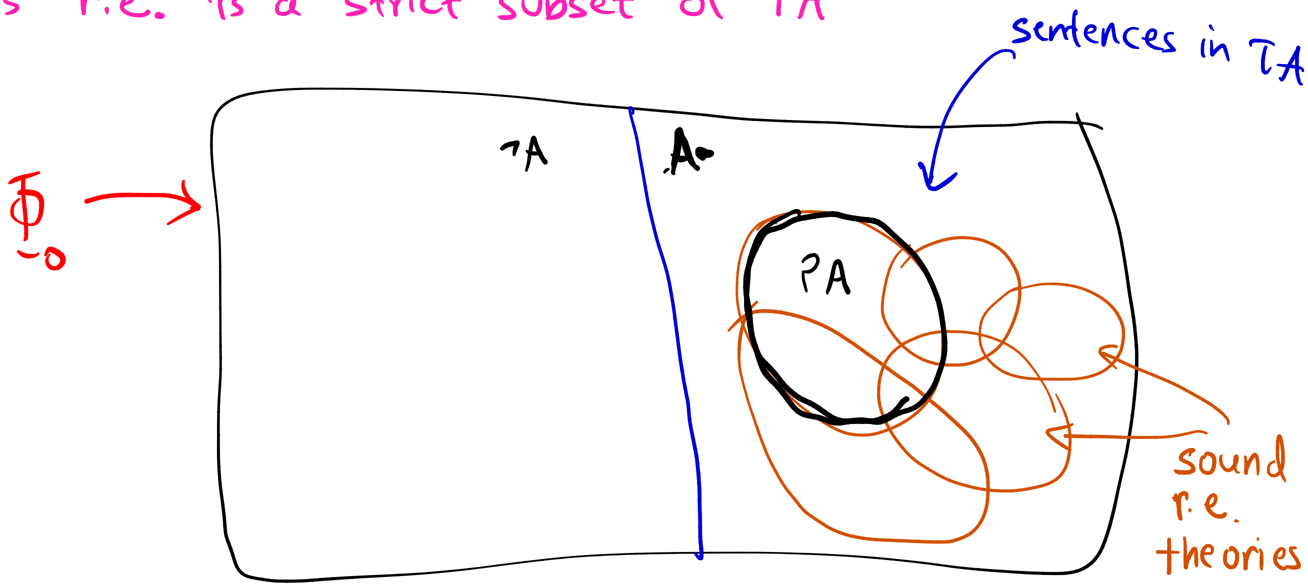
<u>Theorem</u> Σ is axiomatizable iff Σ is r.e.

(p. 76 of Notes)

Incompleteness - Introduction

$\boxed{\text{Incompleteness Theorem of TA:}}$ TA is Not axiomatizable

In other words, any sound theory Σ (sound: $\Sigma \subseteq TA$) that is r.e. is a strict subset of TA



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$K = \{x \mid \{x\} \text{ halts on } x\}$

PROOF:

MAIN LEMMA: For all x there is a sentence F_x (over $\mathcal{L}_A$) such that $x \in K^c$ iff $F_x \in TA$

$\therefore$ If TA is r.e. then K^c is re (contradiction):

Assume TA is r.e. and let M be a TM s.t. $\mathcal{L}(M) = TA$

TM for K^c [given x: Run M on F_x and accept iff $M(F_x)$ accepts

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PROOF:

$\boxed{\text{MAIN LEMMA: For all } x \text{ there is a sentence } F_x \text{ (over } \mathcal{L}_A) \text{ such that } x \in K^c \text{ iff } F_x \in TA}$

Need to show we can reason about TM computations with formulas in $\mathcal{L}_A$

(FIRST) INCOMPLETENESS THEOREM

MAIN LEMMA: For all x there is a sentence F_x (over $\mathcal{L}_A$) such that $x \in K^c$ iff $F_x \in TA$

Defn A predicate is arithmetical if it can be represented by a formula over $\mathcal{L}_A$

EXISTS-DELTA THEOREM (pp 68-71):

Every r.e. predicate/language is arithmetical

$\therefore$ the complement of an r.e. language is arithmetical so in particular K^c is arithmetical

Every R.e. predicate is arithmetical

Definition Let $s_0 = 0$, $s_1 = S0$, $s_2 = SS0$, etc.

Let $R(x_1 .. x_n)$ be an n-ary relation $R \subseteq \mathbb{N}^n$

Let $A(x_1, .., x_n)$ be an $\mathcal{L}_A$ formula, with free variables $x_1, .., x_n$

$A(\vec{x})$ represents R iff $\forall \vec{a} \in \mathbb{N}^n \left[R(\vec{a}) \Longleftrightarrow \mathbb{N} \models A(s_{a_1}, s_{a_2} .. s_{a_n}) \right]$

Example $R \subseteq \mathbb{N}$ $R = \{ a \in \mathbb{N} \mid a \text{ is even} \}$

$A(x) \stackrel{d}{=} \exists y \, (y + y = x)$ $\qquad \exists y \leq x \, (y + y = x)$

$R(3) = \text{false}$ and $\mathbb{N} \not\models A(SSS0) = \exists y \, (y + y = SSS0)$

$R(4) = \text{true}$, and $\mathbb{N} \models A(SSSS0) = \exists y \, (y + y = SSSS0)$

Every R.e. predicate is arithmetical

Definition Let $s_0 = 0$, $s_1 = S0$, $s_2 = SS0$, etc.

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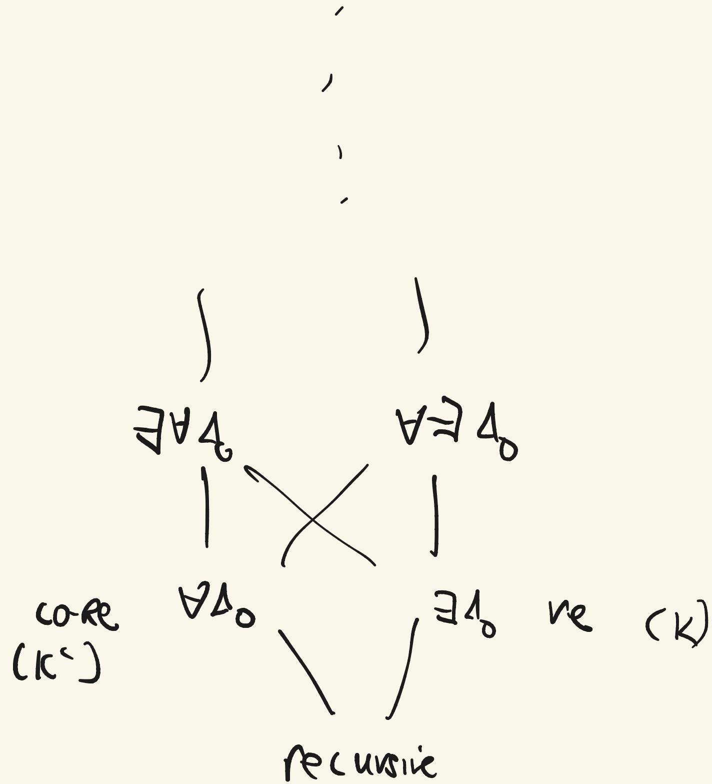
Let $A(x_1, \ldots, x_n)$ be an $\mathcal{L}_A$ formula, with free variables $x_1, \ldots, x_n$

$A(\vec{x})$ represents R iff $\forall \vec{a} \in \mathbb{N}^n \quad R(\vec{a}) \iff \mathbb{N} \models A(s_{a_1}, s_{a_2} \ldots s_{a_n})$

R is arithmetical iff there is a formula $A \in \mathcal{L}_A$ that represents R

Exists-Delta-Theorem every r.e. relation is arithmetical. In fact every r.e. relation is represented by a $\exists\Delta_0$ $\mathcal{L}_A$-formula.

Arithmetic hierarchy



$\exists\Delta_0$ Formulas

$t_1 \le t_2$ stands for $\exists w(t_1 + w = t_2)$

$\left.\begin{array}{lll} \exists z \le t\ A & \text{stands for} & \exists z\,(z \le t \wedge A) \\ \forall z \le t\ A & \text{stands for} & \forall z\,(z \le t \supset A) \end{array}\right\}$ Bounded Quantifiers

Definition A formula is a Δ_0-formula if it has the form

$$\underbrace{\forall z_1 \le t_1\, \exists z_2 \le t_2\, \forall z_3 \le t_3 \ldots \exists z_k \le t_k}_{\text{Bounded Quantifiers}}\ \underbrace{A(\vec{x}, \vec{z}\,)}_{\text{No quantifiers}}$$

Definition A relation $R(\vec{x})$ is a Δ_0-relation iff some Δ_0-formula represents it

$\exists\Delta_0$ Formulas

<u>Example</u> $\text{Prime} = \{x \in \mathbb{N} \mid x \text{ is prime}\}$ is a Δ_0-relation, represented by the following Δ_0-formula:

$$A(x) \overset{d}{=} s0 < x \wedge \forall z_1 \leq x \, \forall z_2 \leq x \, (x = z_1 \cdot z_2 \supset (z_1 = 1 \vee z_1 = x))$$

$$\forall z_1 \leq x \, \forall z_2 \leq \cancel{x} \left((s0 < x) \wedge (x = z_1 \cdot z_2 \supset (z_1 = 1 \vee z_1 = x)) \right)$$

$\exists\Delta_0$ Formulas

$t_1 \le t_2$ stands for $\exists w\,(t_1 + w = t_2)$

$\exists z \le t\ A$ stands for $\exists z\,(z \le t \wedge A)$

$\forall z \le t\ A$ stands for $\forall z\,(z \le t \supset A)$

Bounded Quantifiers

Definition A formula is a Δ_0-formula if it has the form $\forall z_1 \le t_1\, \exists z_2 \le t_2\, \forall z_3 \le t_3 \ldots \exists z_k \le t_k\, A(\vec{x}, \vec{z})$

Definition A $\exists\Delta_0$ formula has the form $\exists \vec{y}\, \underbrace{B(\vec{x}, \vec{y}, \vec{z})}_{\Delta_0 \text{ formula}}$

Definition A relation $R(\vec{x})$ is a Δ_0-relation iff some Δ_0-formula represents it

Definition $R(\vec{x})$ is a $\exists\Delta_0$-relation iff some $\exists\Delta_0$-formula represents it

$\exists\Delta_0$ Formulas

Lemma Every Δ_0 relation is recursive

Lemma Every $\exists\Delta_0$ relation is r.e.

$\exists\Delta_0$ (Exists-Delta) Theorem every r.e. relation is represented by a $\exists\Delta_0$ formula

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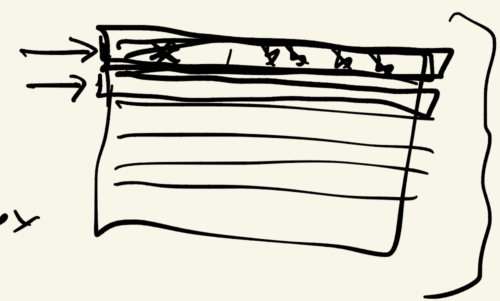
$K = \{x \mid \{x\} \text{ halts on } x\}$

$A: \underbrace{\exists y}_{\text{tableaux of TM } \{x\} \text{ running on } x}$ "y describes tableaux of TM $\{x\}$ on input x & final line of tableaux halts !!"



$\exists\Delta_0$ Theorem

Main Lemma Let $f : \mathbb{N}^n \to \mathbb{N}$ be a total computable function.

Let $R_f = \{ (\vec{x}, y) \in \mathbb{N}^{n+1} \mid f(\vec{x}) = y \}$ ← also called graph(f)

Then R_f is a $\exists\Delta_0$-relation.

Main Lemma Let $f: \mathbb{N}^n \to \mathbb{N}$ be total, computable

Then $R_f = \{ (\vec{x}, y) \mid f(\vec{x}) = y \}$ is a $\exists\Delta_0$ relation

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Proof of $\exists\Delta_0$ Theorem from Main Lemma

Let $R(\vec{x})$ be an r.e. relation [example $K(x)$] $\quad K(x) = \exists y\, S(x,y)$

Then $R(\vec{x}) = \exists y\, S(\vec{x}, y)$ where S is recursive

Since S is recursive, $f_s(\vec{x}, y) = \begin{cases} 1 & \text{if } (\vec{x}, y) \in S \\ 0 & \text{otherwise} \end{cases}$

is total computable

By main lemma R_{f_s} is represented by a $\exists\Delta_0$ relation

So $R(\vec{x}) = \exists y \underbrace{\exists z\, B}_{R_{f_s}}$ is represented by a $\exists\Delta_0$ relation

Let $K = \{x \mid \{x\} \text{ halts on input } x\}$

Can describe K by:

$$K = \exists y \underbrace{A(x,y)}_{\exists \bar{z}\, \underbrace{F(x,y,\bar{z})}_{\Delta_0}}$$

where A is the recursive relation that accepts iff y is the tableaux of TM $\{x\}$ when run on input x and last config of y halts

A is recursive so by main lemma A is represented by an $\exists\Delta_0$ formula

$\therefore$ K is also represented by $\exists\Delta_0$ formula

Proof of Main Lemma: MAIN IDEA

Let $f : \mathbb{N} \to \mathbb{N}$ be unary, total computable function, & let M_f be TM computing f

$R(\vec{x}, y)$ will be a $\exists\Delta_0$ relation saying:

$\exists m, \underbrace{c, d}$ such that

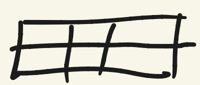
(1) c, d describe the tableaux given by $r_1 \ldots r_m \ldots r_{m^2}$

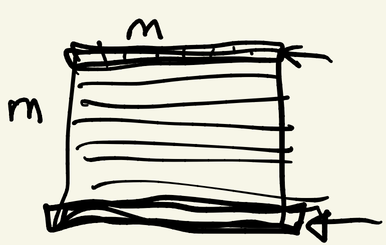
(2) $r_1 \ldots r_m$ encode start config of M_f on x

(3) Last m numbers $r_{(m-1)m} \ldots r_{m^2}$ encode last config, containing y in first cells then B, and state is q_z

(4) For all other configs, state is not q_z.

(5) all 2×3 local cells are consistent with transition function of M_f





Proof of Main Lemma: MAIN IDEA

Let $f: \mathbb{N} \to \mathbb{N}$ be unary, total computable function, + let M_f be TM computing f

$R(\vec{x}, y)$ will be a $\exists \Delta_0$ relation saying:

$A(x,y)$ formula

$\exists m, c, d$ such that

(1) c, d describe the tableaux given by $r_1 \dots r_m \dots r_{m^2}$

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(4) For all other configs, state is not q_z.

(5) all 2×3 local cells are consistent with transition function of M_f

- Need to encode an arbitrarily long sequences (of Numbers/strings) by a few (3) numbers (m, c, d)
- Need formulas that can talk about the i^{th} number in the sequence

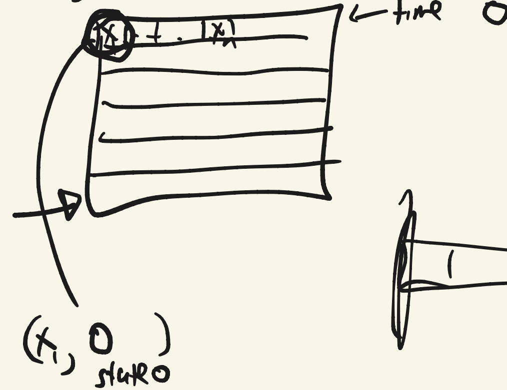
We want $A(x,y)$ to be true iff

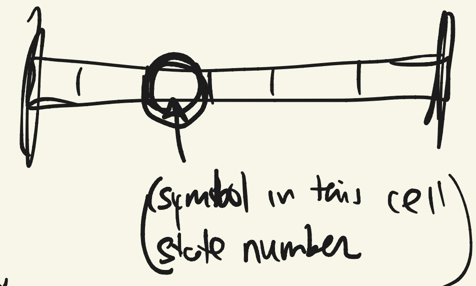
M_f on input x halts + outputs y

iff $\exists\, m$ (runtime of M_f on x)

$\exists$ tableaux T

m^2 cells

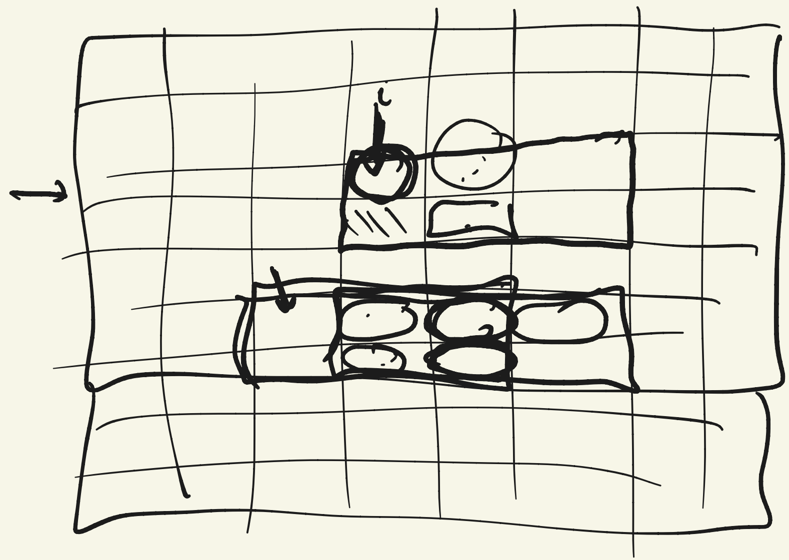




s.t. T is correct:

1. 1st row of T is start config of M_f on x
2. last row of T outputs y + is in halt state
3. No other rows are in halt state
4. for every row $i>1$, row i is the config of M_f on x after i steps

4th condition can be checked locally



Proof of Main Lemma: MAIN IDEA

- Need to encode an arbitrarily long sequences (of Numbers/strings) by a few (3) numbers (m, c, d)
- Need formulas that can talk about the i^{th} number in the sequence
- WARMUP: if exponentiation fxn x^y were in $\mathcal{L}_A$, this would be easier.

encode 57, 3009, 205, 4, 5 by

$$2^{57} \cdot 3^{3009} \cdot 5^{205} \cdot 7^{4} \cdot 11^{5}$$

(ie i^{th} number x sequence encoded by p_i^x, where $p_i = i^{th}$ smallest prime number)

Proof of Main Lemma: MAIN IDEA

- Need to encode an arbitrarily long sequences (of Numbers/strings) by a few (3) numbers (m, c, d)
- Need formulas that can talk about the i^{th} number in the sequence
- WARNUP: if exponentiation fxn x^y were in $\mathcal{L}_A$, this would be easier.

- But we need to encode sequences using <u>only</u> $+, \cdot, s$

  ★ gödel's β function does this using magic of chinese remainder theorem

Proof of Main Lemma (see pp 70-71)

Main idea: is a way of representing sequences of numbers by numbers using $\exists\Delta_0$ formulas

Note: Prime power decomposition not useful here since we only have $s, +, \cdot$
(ie. represent (a_1, a_2, a_3, a_4) by $2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot 7^{a_4}$)

Definition β-function

$$\beta(c, d, i) = rm(c, d(i+1)+1)$$

where $rm(x, y) = x \bmod y$

Proof of Main Lemma (see pp 70-71)

Definition β-function

$$\beta(c, d, i) = rm(c, d(i+1)+1) \quad \text{where } rm(x, y) = x \bmod y$$

Lemma 0. $\forall n, r_0, r_1, \ldots, r_n \; \exists c, d$ such that

$$\beta(c, d, i) = r_i \quad \forall i,\ 0 \le i \le n$$

So the pair (c, d) represents the sequence $r_0 r_1, \ldots r_n$ using β

Proof of Main Lemma (see pp 70-71)

Definition β-function

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CRT (Chinese Remainder Theorem)

Let $r_0, \ldots, r_n$, $m_0, \ldots, m_n$ be such that
$0 \le r_i \le m_i \ \ \forall i, \ 0 \le i \le n$ and $\gcd(m_i, m_j) = 1 \ \ \forall i, j$

Then $\exists r$ such that $rm(r, m_i) = r_i \ \ \forall i, \ 0 \le i \le n$

CRT (Chinese Remainder Theorem)

Let $r_0, \dots, r_n$, $m_0, \dots, m_n$ be such that:

(1) $0 \le r_i \le m_i \quad 0 \le i \le n$

(2) $\gcd(m_i, m_j) = 1 \quad \forall i,j \,,\ i \ne j$

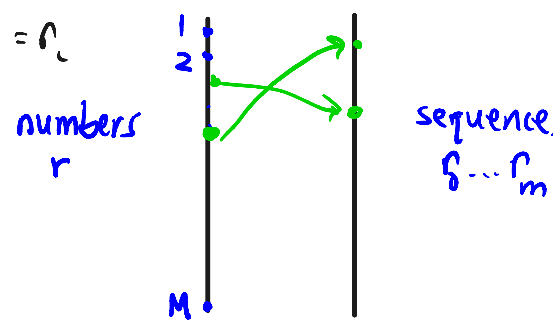
Then $\exists r$ such that $rm(r, m_i) = r_i \quad \forall i,\ 0 \le i \le n$

Proof (counting Argument)

- The number of sequences $r_0 \dots r_n$ such that (1) holds is
  $M = m_0 \cdot m_1 \cdot \ldots \cdot m_n$
- Each r, $0 \le r \le M$ corresponds to a different sequence:

  Ie. If $\forall i\ rm(r, m_i) = r_i$ and $\forall i\ rm(s, m_i) = r_i$
  Then $r = s$ (mapping is 1-1)

  ∴ for every sequene $r_0 \dots r_m$, some $r \le M$ maps to it



Proof of Main Lemma (see pp 70-71)

Lemma b $\forall n, r_0, r_1, \ldots, r_n \ \exists c, d$ such that
$\beta(c,d,i) = r_i \quad \forall i, \ 0 \le i \le n$

$\beta(c,d,i) = rm(c, d(i+1)+1)$
where $rm(x,y) = x \bmod y$

Chinese Remainder Theorem

Let $r_0, \ldots, r_n$, $m_0, \ldots, m_n$ be such that $0 \le r_i \le m_i$ and $\gcd(m_i, m_j) = 1$. Then $\exists r$ $rm(r, m_i) = r_i \ \forall i$

Proof of Lemma b

Let $d = (n + r_0 + \ldots + r_n + 1)!$

Let $m_i = d(i+1)+1$

Claim $\forall i,j$ $\gcd(m_i, m_j) = 1$ (proof next page)

By CRT $\exists r = c$ so that $\beta(c,d,i) = rm(c, m_i) = r_i \ \forall i \in [n]$

<u>Claim</u> Let $d = (n + r_0 + r_1 + \dots + r_n + 1)!$, $m_i = d(i+1)+1$

then $\forall i \neq j \le n$ $\gcd(m_i, m_j) = 1$

<u>Pf</u> Suppose p is a prime, and $p \mid \underbrace{d(i+1)+1}_{m_i}$, $p \mid \underbrace{d(j+1)+1}_{m_j}$

Then $p \mid [\underbrace{d(j+1)+1}_{m_j}] - [\underbrace{d(i+1)+1}_{m_j}]$ (assume $j > i$)

so $p \mid d(j-i)$

But p cannot divide both d and $d(i+1)+1$ so $p \mid j-i$

But then $p \le j-i < n$ so $p \mid d$ #

Proof of Main Lemma (see pp 70-71)

Lemma 0 $\forall n, r_0, r_1, \ldots, r_n \ \exists c, d$ such that
$\beta(c,d,i) = r_i \quad \forall i, \ 0 \le i \le n$

Lemma 1 graph(β) is a Δ_0 relation

Pf

We want a Δ_0 formula $A(z_1 z_2 z_3 z_4)$ s.t.
A is true on inputs c, d, i, y iff $\beta(c,d,i) = y$

$$y = \beta(c,d,i) \iff c \bmod d(i+1)+1 = y$$

$$\iff c = [d(i+1)+1]q + y \qquad \text{where } y < d(i+1)+1$$

$$y = \beta(c,d,i) \iff [\exists q \le c\,(c = q(d(i+1)+1) + y) \wedge y < d(i+1)+1]$$

Proof of Main Lemma (see pp 70-71)

Let $f : \mathbb{N} \to \mathbb{N}$ be unary, total computable function, & let M_f be TM computing f

$R(\vec{x}, y)$ will be a $\exists \Delta_0$ relation saying:

$\exists m, c, d$ such that

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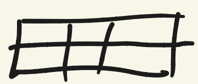


Recap: we wanted to prove

$\exists\Delta_0$ (Exists-Delta) Theorem every r.e. relation is represented by a $\exists\Delta_0$ formula

which followed by **Main Lemma**:

f total, computable $\Rightarrow R_f$ is a $\exists\Delta_0$ relation

Recap:

First Incompleteness Theorem

1st Incompleteness Theorem: TA is NOT axiomatizable

That is, any sound, axiomatizable theory is incomplete.

→ PA is axiomatizable. So assuming PA is sound, it is incomplete (so there are sentences A such that neither A or $\neg A$ is provable from axioms of PA.)

Recap:

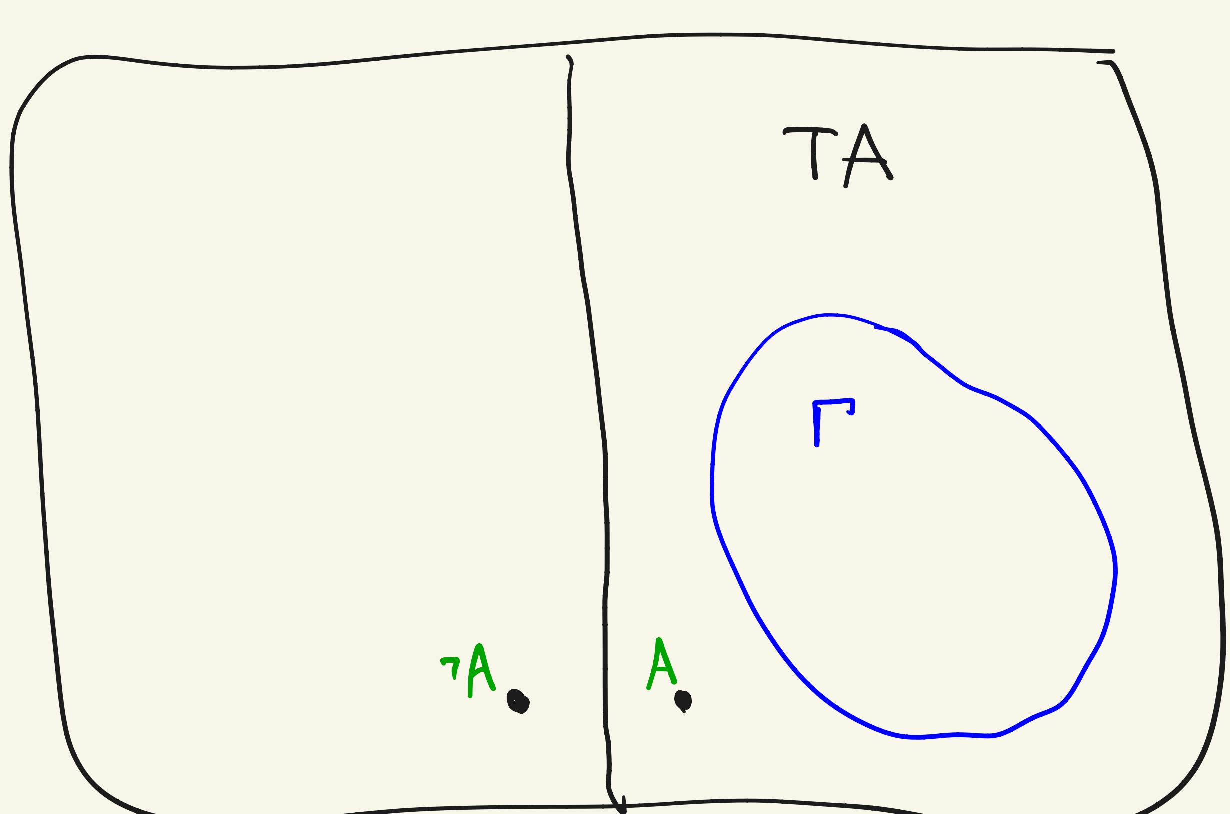
First Incompleteness Theorem

1st Incompleteness Theorem: TA is NOT axiomatizable

That is, any sound, axiomatizable theory is incomplete.

→ PA is axiomatizable. So assuming PA is sound, it is incomplete (so there are sentences A such that neither A or $\neg A$ is provable from axioms of PA.)

Φ_0: all $\mathcal{L}_A$ sentences



Γ sound and axiomatizable $\Rightarrow \exists A, \neg A \notin \Gamma$

Tarski Theorem

Define the predicate Truth $\subseteq \mathbb{N}$

$$\text{Truth} = \{ n \mid n \text{ encodes a sentence } \langle n \rangle \in TA \}$$

Then Truth is not arithmetical.

By $\exists\Delta_0$-Theorem (every r.e. set/language is arithmetical) this implies that Truth is not r.e.

High Level idea of Proof:

Formulate a sentence "I am false" which is self-contradictory

Pf of Tarski's Thm

Let $\text{sub}(m,n) = \begin{cases} 0 \text{ if } m \text{ is not a legal encoding of a formula} \\ \text{otherwise say } m \text{ encodes the formula } A(x) \text{ with free variable } x. \text{ Then } \text{sub}(m,n) = m' \text{ where } m' \text{ encodes } A(s_n) \end{cases}$

[sub(m,n): decode m, plug in n + re-encode]

Let $d(n) = \text{sub}(n,n)$

$\begin{cases} d(n) = 0 \text{ if } n \text{ not a legal encoding.} \\ \text{ow say } n \text{ encodes } A(x). \\ \text{then } d(n) = n' \text{ where } n' \text{ encodes } A(s_n) \end{cases}$

clearly sub, d are both computable

so by $\exists \Delta_0$-theorem graph(sub), graph(d) are arithmetical

Proof of Tarski's Thm

Suppose that Truth is arithmetical.
Then define $R(x) = \neg \mathrm{Truth}(d(x))$

Since d, Truth both arithmetical, so is R

Let $\widetilde{R(x)}$ represent $R(x)$, and let e be the encoding of $\widetilde{R(x)}$

Let $d(e) = \underbrace{\widetilde{R(s_e)}}$

encodes "I am false"

Then

$$\widetilde{R(s_e)} \in TA \iff \neg \mathrm{Truth}(d(e)) \quad \text{since } \tilde{R} \text{ represents } R$$

$$\iff \neg \widetilde{R(s_e)} \in TA \quad \text{by defn of truth}$$

$$\iff \widetilde{R(s_e)} \notin TA \quad TA \text{ contains exactly one of } A, \neg A$$

↯ this is a contradiction. ∴ Truth is not arithmetical

PEANO ARITHMETIC

$$\begin{array}{ll}
\text{P1.} & \forall x\,(sx \neq 0) \\
\text{P2.} & \forall x\,\forall y\,(sx = sy \supset x = y) \qquad s \text{ is } 1\text{-}1 \\
\text{P3.} & \forall x\,(x + 0 = x) \\
\text{P4.} & \forall x\,\forall y\,(x + sy = s(x+y))
\end{array}
\quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{define } +$$

$$\left.\begin{array}{ll}
\text{P5.} & \forall x\,(x \cdot 0 = 0) \\
\text{P6.} & \forall x\,\forall y\,(x \cdot sy = (x \cdot y) + x)
\end{array}\right\} \text{define } \cdot$$

$$\mathrm{IND}(A(x)) : \forall y_1 \ldots \forall y_k \left[\left(A(0) \wedge \forall x\,(A(x) \supset A(sx)) \right) \supset \forall x\, A(x) \right]$$

INDUCTION AXIOMS: All sentences $\mathrm{IND}(A(x))$ for all formulas A whose free variables are $y_1 \ldots y_k, x$

$$\Gamma_{PA} = \{P_1, \ldots, P_6\} \cup \{\text{INDUCTION AXIOMS}\}$$

1. Γ_{PA} is recursive

2. PA is sound & axiomatizable (so incomplete)

3. PA still strong enough to prove all of standard number theory

Robinson's Arithmetic RA

Axioms $\{P1, \ldots, P6\}$ of PA plus P7, P8, P9

P7 : $(\forall x \ x \leq 0 \supset x = 0)$

P8 : $\forall x \forall y \, (x \leq sy \supset (x \leq y \vee x = sy))$

P9 : $\forall x \forall y \ (x \leq y \vee y \leq x)$

where $t_1 \leq t_2$ abbreviates $\exists z (t_1 + z = t_2)$

FACTS ① $RA \subseteq PA$

② RA finitely axiomatizable

Stronger Version of Incompleteness Thm

Recall

$R(\vec{x})$ is represented by an $\exists\Delta_0$ formula $A(\vec{x})$ if

$$\forall \vec{a} \in \mathbb{N} \quad R(\vec{a}) \Leftrightarrow TA \models A(S_{\vec{a}})$$

Stronger version:

$R(\vec{x})$ is represented in RA by $A(\vec{x})$ if

$$\forall \vec{a} \in \mathbb{N} \quad R(\vec{a}) \Leftrightarrow RA \models A(S_{\vec{a}})$$

RA Representation Theorem

Every r.e. relation is represented in RA by an $\exists\Delta_0$ formula

Corollaries of RA Representation Theorem

1. RA is Not recursive

Pf sketch: K is r.e. but Not recursive

K r.e. $\Rightarrow$ it is represented in RA by some $\exists\Delta_0$-formula A

If RA recursive then K recursive. Contradiction

2. VALID is not recursive

Pf idea: RA is finitely axiomatizable!

$A \in RA \iff P1 \wedge \dots \wedge P9 \Rightarrow A$ is valid

so membership in RA is reducible to membership in VALID

RA Representation Theorem

Every r.e. relation is represented in RA by an $\exists\Delta_0$ formula

Proof idea

$\boxed{\text{Main Lemma}}$: every Δ_0-sentence in TA is provable in RA

Assuming Main Lemma, Let $R(\vec{x})$ be an r.e. relation.

By Exists-Delta Theorem, $R(\vec{x})$ is represented (in TA) by some $\exists\Delta_0$-formula $A(\vec{x})$

so $\forall \vec{a} \in \mathbb{N}^k \quad R(\vec{a}) \iff \exists y \underbrace{A(S_{\vec{a}}, y)}_{\Delta_0} \in TA$

$\therefore$ By soundness of RA & since every $\exists\Delta_0$ sentence of TA is provable in RA

$$R(\vec{a}) \iff RA \vdash \exists y\, A(S_{\vec{a}}, y)$$

So $\exists y\, A(\vec{x}, y)$ represents $R(\vec{x})$ in RA

2nd INCOMPLETENESS THEOREM

Recall PA is a strong sound subtheory of TA

2nd Incompleteness Thm

PA cannot prove its own consistency

2nd Incompleteness Thm

(1) A specific sentence "I am not provable" $\equiv g$ such that neither g nor $\neg g$ are provable in PA (assuming PA is consistent)

(2) Consistency of PA, Con(PA) is not provable in PA (assuming PA is consistent)

- Let Γ_{PA} be the set of axioms of PA
- Let $\mathrm{Proof}(x,y)$: true if and only if y codes a LK-Γ_{PA} proof of the sentence coded by x
- Recall $d(n) = \#A(s_n)$ where $\#A(x) = n$
  (so n codes the formula $A(x)$, and $d(n)$ codes $A(s_n)$)

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  (so n codes the formula $A(x)$, and $d(n)$ codes $A(s_n)$)
- Let $S(x)$ be the r.e. relation: $\exists y\, Proof(d(x), y)$
- By RA representation Theorem, let $A(x)$ be a $\exists\Delta_0$ formula that represents $S(x)$ in RA (& hence in PA)
- Then $\forall n \in \mathbb{N}$ $\quad \exists y\, Proof(d(n), y) \Longleftrightarrow PA \vdash A(s_n) \quad (*)$

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  (so n codes the formula $A(x)$, and $d(n)$ codes $A(s_n)$)

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- By RA representation Theorem, let $A(x)$ be a $\exists\Delta_0$ formula that represents $S(x)$ in RA (& hence in PA)

- Let $e = \#\neg A(x)$, so $d(e) = \#\neg A(s_e)$

- Let $g \overset{d}{=} \neg A(s_e)$

  says that "I am not provable"
  since $\neg A(s_e)$ says the formula encoded by $d(e)$ -- which is g -- is not provable in PA

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- Let $g \stackrel{d}{=} \neg A(S_e)$

Theorem PA consistent $\Rightarrow$ PA $\nvdash g$

PF Suppose PA $\vdash g$

Then sentence number $d(e)$ is provable, so $\exists y \, \text{Proof}(d(e), y)$ holds

Thus PA $\vdash A(S_e)$ by left-to-rt direction of (*)

Thus PA $\vdash \neg g$ and PA $\vdash g$ so PA not consistent

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- By RA representation Theorem, let $A(x)$ be a $\exists\Delta_0$ formula that represents $S(x)$ in RA (& hence in PA)
- Let $e = \# \neg A(x)$, so $d(e) = \# \neg A(S_e)$
- Let $g \overset{d}{=} \neg A(S_e)$

Theorem PA consistent $\Rightarrow$ PA $\nvdash \neg g$

<u>Pf</u> Suppose PA $\vdash \neg g$. ie PA proves $A(S_e)$

Then $\exists y \, Proof(d(e), y)$ by rt-to-left direction of (*)

So PA proves $\neg A(S_e)$

So PA $\vdash g$ and PA $\vdash \neg g$, so PA not consistent

Formulating consistency in PA

Let $B(x,y)$ be a $\exists\Delta_0$ formula that represents
Proof(x,y) in RA (and thus also in PA)

Then for every sentence C

$$PA \vdash C \iff PA \vdash \exists y \underbrace{B(\#C, y)}_{\text{stands for } B(S_{\#C}, y)}$$

Then $PA \vdash A(S_n) \supset \exists y\, B(S_{d(n)}, y)$

[recall $A(x)$ represents $\exists y\, B(d(x), y)$]

Formulating consistency in PA

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[recall $A(x)$ represents $\exists y\, B(d(x), y)$]

Define $con(PA) \overset{d}{=} \neg \exists y\, B(\#0\neq 0, y)$

Theorem If PA is consistent, then $PA \nvdash con(PA)$

Proof:

Main Lemma: $PA \vdash (con(PA) \supset g)$

[recall $g \stackrel{d}{=} \neg A(s_e)$, $e = \# \neg A(x)$ says "I am not provable"]

If $PA \vdash con(PA)$ by main Lemma $PA \vdash g$

But by previous theorem
PA consistent $\Rightarrow PA \nvdash g$

$\therefore$ PA consistent $\Rightarrow PA \nvdash con(PA)$

It is left to prove:

Main Lemma: $PA \vdash con(PA) \supset g$

[recall $g \overset{d}{=} \neg A(s_e)$, $e = \# \neg A(x)$ says "I am not provable"]

↑

Need to formalize Proof of Gödel's Incompleteness Thm in PA. Main step is to formalize in PA that every true $\exists \Delta_0$ sentence is provable in RA.