

Homework 4

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Due: Nov 13, 2025 at 11am

Collaboration: Collaboration with other students in the class homework is allowed. However, you must write up solutions by yourself and understand everything that you hand in. You may also consult other reference materials, but you may not seek out answers from other sources or use AI tools. **You are required to list who you collaborated with on each problem, including any TAs or students you discussed the problems with in office hours. Also list any reference materials consulted other than the lectures and textbook for our class. See the course webpage for more details.**

Formatting: Write the solution to each part of each problem on a separate page. Be sure to correctly indicate which page each problem appears on in GradeScope. Please do not write your name on any page of the submission; we are using anonymous grading in GradeScope. If we can't find your solution to any problems because it was not properly tagged with the page, or if handwriting is not legible, you will receive 20 percent on these problems.

Grading: There are 74 total possible points, not counting the last Bonus problem. For all problems except for the Bonus problem, you have the option of answering “Don't know” on any parts of the question, and you will receive 20 percent of the total marks for those parts. *You will also be graded on clarity and brevity of your answers. For each question below, we have added guidelines for the length of your solution.*

0 Exercises

Here are some **highly** recommended exercises. You should not turn in your solutions for these, and we will not grade them.

(a) Prove that for every infinite set S , the following are equivalent:

- (i) There exists a function $g : \mathbb{N} \rightarrow S$ that is onto (i.e., g is surjective).
- (ii) There exists a function $f : S \rightarrow \mathbb{N}$ that is one-to-one (e.g., f is injective).

(b) Are the following languages (i) decidable or (ii) undecidable. Prove your answer.

- $L = \{\langle M \rangle \mid M \text{ is a TM and for every even length } \ell, \text{ there is a string } w, |w| = \ell, \text{ such that } M \text{ halts on } w\}$
- $L = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is finite and has size divisible by 3} . \}$

(c) Prove that L_{pug} (as defined in the bonus problem) is undecidable.

1 Turing Machines (20 points)

Consider the following two Turing Machines M_1 and M_2 . If a transition isn't drawn, it goes to the reject state. The accept state is q_{acc} . We use B to represent a blank ($\sqcup$ in the book).

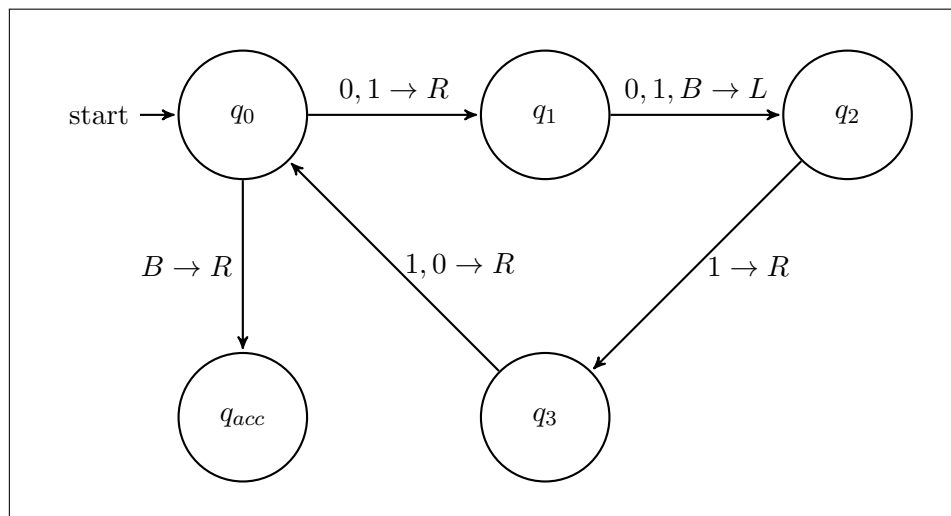


Figure 1: The Turing Machine M_1

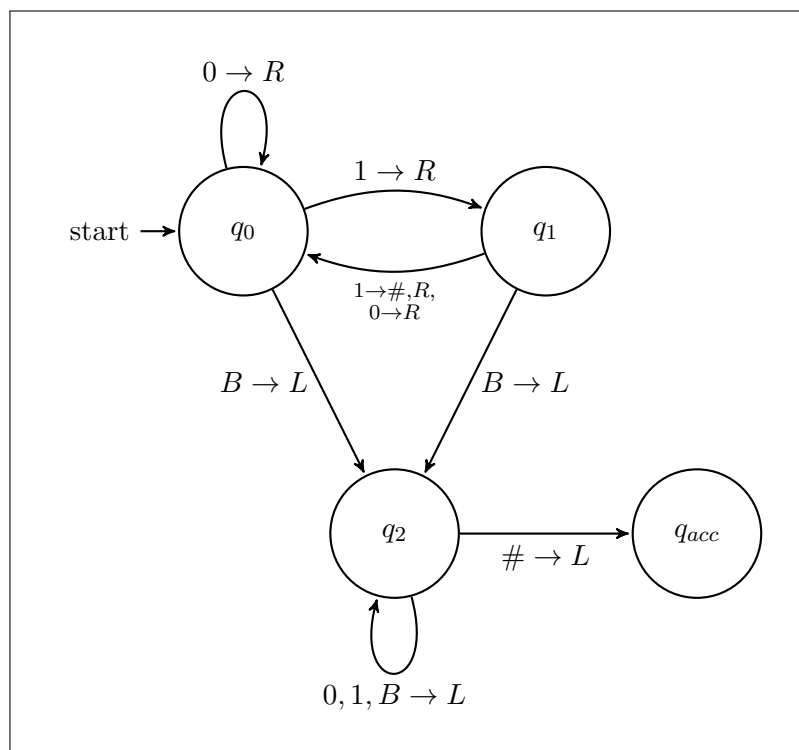


Figure 2: The Turing Machine M_2

Answer each of the following. No justification required.

- (2 points) True or False ? The Machine M_1 always halts.
- (6 points) What is $L(M_1)$, give your answer using set notation. ¹
- (2 points) True or False ? $L(M_1)$ is decidable.
- (2 points) True or False ? The Machine M_2 always halts.
- (6 points) What is $L(M_2)$, give your answer using set notation.
- (2 points) True or False ? $L(M_2)$ is decidable.

To answer these questions, it is worth trying to run the TM on a few short strings and try to see the pattern. ²

¹For instance $L(M_1) = \{w \mid w \text{ starts with } 11 \text{ and doesn't contain } 00\}$.

²You can use websites such as this one.

Instructions for problems 2 to 5

To prove a language L is recognizable (or decidable), you must give a **high-level** description of a TM that accepts L and proof it is correct. See the last page of the HW for what we expect when we mean **high-level** description and what we expect for the proof of correctness.

For each question, you can ignore inputs that are *bad encodings*. For instance in Problem 3, you can ignore inputs that are not of the form $\langle M \rangle$.

When proving something is not decidable, make sure you write a *proof* (See Chapter 5 in the book for examples). You are not allowed to use Rice's theorem in any of these problems. If you want to use the fact some language L is not decidable/recognizable, we must have seen this language in class³

2 Streaming “Algorithms” (10 points)

Before introducing Turing Machines, we introduced streaming algorithms (see the lecture notes for the definition). Is the following statement true? Give a proof or disprove it.

If $L \subseteq \Sigma^$ has a streaming algorithm⁴, then L is decidable.*

Note: There are many ways to prove (or disprove) the above statement. Make sure that you give a *proof*.

3 Decidability/Recognizability of L_{pair} (24 points, 12 points each)

Let

$$L_{\text{pair}} := \{ \langle M \rangle \mid M \text{ is a TM that accepts two consecutive binary numbers} \}.$$

For example, if a Turing machine M accepts both 100 and 101 then $\langle M \rangle \in L_{\text{pair}}$.

- a) Prove that L_{pair} is recognizable.
- b) Prove that L_{pair} is not decidable.

4 Recognizable versus Decidable (20 points, 10 points each)

Classify each of the following languages as either: (i) decidable or (ii) undecidable. *Prove* your answers.

- a) $L = \{ \langle M, w \rangle \mid M \text{ is a TM, } w \text{ a string and } M \text{ moves its head left at least once on input } w \}$.
- b) $L = \{ \langle M, w, q \rangle \mid M \text{ enters state } q \text{ on input } w \text{ (So, } M \text{ is a TM, } q \text{ a state of } M \text{ and } w \text{ a string)} \}$.

³for instance you're allowed to use A_{TM} or $HALT_{TM}$.

⁴I.e. there is a streaming algorithm $\mathcal{A}$ with $L(\mathcal{A}) = L$

5 Bonus Problem

We say that a Turing Machine M is *pugnacious* if it has all of the following properties:

- It accepts the string 1111111111 and the string 1010101010
- It rejects the string 000111000111 and the string 000000000000
- It loops (never accepts or rejects) on the string 0101010101 and the string 111000111000

Define the language L_{pug} by

$$L_{pug} := \{\langle M \rangle \mid M \text{ is a pugnacious Turing Machine}\}.$$

- (a) (up to 20 additional points) Your goal in this problem is to give the code for a pugnacious Turing Machine using the following website: <https://turingmachinesimulator.com/>. Be sure to look at some examples and tutorials on that site to learn about the notation it uses.

If your Turing Machine is not Pugnacious, or if it doesn't compile on the website as a **1-tape** Turing Machine, then you will get 0 points. Otherwise, we call your submission “valid”, and your score on this problem will be calculated as follows.

Let n be the number of students in the class who make a valid submission. Let k be the number of students in the class who make a valid submission that uses at least as many total lines of code as yours. Then your score will be

$$5 + \left\lceil \frac{15k}{n} \right\rceil.$$

In other words, you want to use as few lines of code as possible, and you will get more points if you use fewer lines of code than other students. Good luck!

(We will post instructions for submitting your code as a text file closer to the deadline. We will measure “total lines of code” as the number of lines in the file you submit.)

- (b) (up to 3 points) In light of part a, it may be difficult for the course staff to grade part b, even if they used the website <https://turingmachinesimulator.com/> or an AI tool like ChatGPT. Explain what the apparent issue is (we are looking for a specific, technical answer here), and give a suggestion for how the course staff might be able to grade it anyway. Give 2 sentences max.

You are not allowed to use “IDK” on the bonus problems.

High Level Description: You do not need to discuss the mechanics of an actual TM (tapes, states, the transition function). *Instead you need present your algorithm in pseudocode.* See chapter 5 of the book for examples. Make sure your algorithm is clear and well defined (e.g., what is the input to the program, what are the main subroutines/loops/steps; specify the data structure(s) being used (variables, arrays, etc). You can give a few sentences to describe the ideas behind your algorithm, **but we expect your algorithm to be pseudocode (do not give us a big paragraph of text of as your algorithm).**

You need to give 3 to 5 sentences proof of why your TM is correct.

- If you are showing that L is recognizable, you need to explain why your TM accepts every string in L . And why it doesn't accept (loop or rejects) any string not in L .
- If you are showing L is decidable, you need to explain why your TM accepts every string in L . And why it rejects any string not in L .