CS Theory (Fall 25)

Midterm Review Solutions

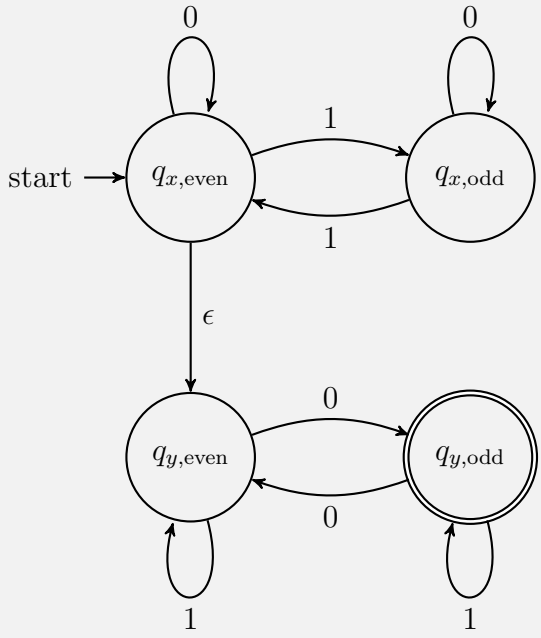
DFA/NFA

Construct a DFA (or NFA) for the following languages and explain why your DFA (or NFA) is correct.

1. $L_1 = \{xy \mid x \in \{0,1\}^* \text{ has an even number of 1's and } y \in \{0,1\}^* \text{ has an odd number of 0's}\}$

Solution.

The following NFA recognizes L_1:



Proof. For any string $w \in L_1$, it can be written as $w = xy$ where x has an even number of 1's and y has an odd number of 0's. The NFA starts in state $q_{x,\text{even}}$ and processes x.

- It stays in the top two states, switching between them whenever it reads a 1, and staying in the same state whenever it reads a 0. Since x has an even number of 1's, the NFA will end up back in state $q_{x,\text{even}}$ after reading all of x.
- Then, the NFA can take the ϵ-transition to state $q_{y,\text{even}}$ and process y.
- It stays in the bottom two states, switching between them whenever it reads a 0, and staying in the same state whenever it reads a 1. Since y has an odd number of 0's, the NFA will end up in state $q_{y,\text{odd}}$, which is an accepting state, after reading all of y.

Thus, the NFA accepts w.
Then we prove the converse. Suppose w is accepted by some computation path of the NFA. Call $\{q_{x,\text{even}}, q_{x,\text{odd}}\}$ the x-stage and $\{q_{y,\text{even}}, q_{y,\text{odd}}\}$ the y-stage. Since

- The NFA starts in x-stage,
- the only accepting state is in the y-stage,
- the only transition edge from the x-stage to the y-stage is an ϵ-transition, no other ϵ-transitions in the NFA,
- no transition edges from the y-stage back to the x-stage,

we conclude that the computation path must contain exactly one ϵ-transition. This ϵ-transition splits the computation path into two parts: the x-part before the ϵ-transition and the y-part after the ϵ-transition.

- In the x-part, the string associated with the computation path must have an even number of 1's since the NFA both starts and ends in state $q_{x,\text{even}}$.
- In the y-part, the string associated with the computation path must have an odd number of 0's since the NFA starts in state $q_{y,\text{even}}$ and ends in state $q_{y,\text{odd}}$.

Thus, w can be written as $w = xy$ where x has an even number of 1's and y has an odd number of 0's (corresponding to the path in x-part and y-part), i.e. $w \in L_1$. □

2. $L_2 = \{w \in \{0,1\}^* \mid w \in L \text{ and the length of } w \text{ is a multiple of 2 or 3}\}$, where L is a regular language recognized by the DFA $(Q, \Sigma, \delta, q_0, F)$.

Solution.

We construct a DFA $(Q', \Sigma, \delta', q_0', F')$ as follows:

- The states are $Q' = Q \times \{0,1,2,3,4,5\}$ (remembering the current string length mod 6).
- The start state is $q_0' = (q_0, 0)$.
- For any $(q,i) \in Q'$ and $a \in \Sigma$, the transition function is

$$\delta'((q,i),a) = (\delta(q,a), (i+1) \bmod 6).$$

- The accepted states $F' \subseteq Q'$ are those (q,i) such that $q \in F$ and $i \in \{0,2,3,4\}$.

This DFA decides L_2.

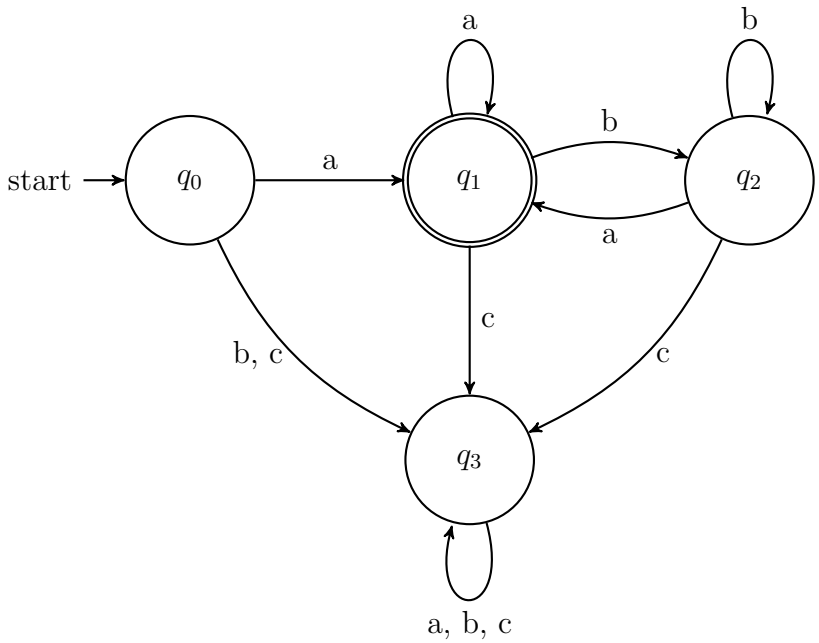
Proof. Consider any string $w \in \Sigma^*$. By mathematical induction on the length of w, we can show that after reading w, the DFA is in state (q,i) where

- State q is the state of the original DFA after reading w.
- Index i is $|w| \bmod 6$, where $|w|$ is the length of w.

Whether "$|w|$ is a multiple of 2 or 3" is completely determined by its residue class mod 6. Specifically, $|w|$ is a multiple of 2 or 3 if and only if $|w| \bmod 6 \in \{0,2,3,4\}$. Thus, w is accepted by the new DFA if and only if w is accepted by the original DFA and $|w|$ is a multiple of 2 or 3, i.e. $w \in L_2$. □

Describe the corresponding regular language for the following DFAs or NFAs.

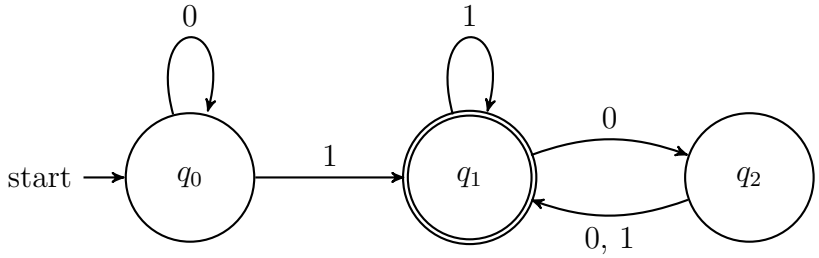
1.



Solution.

The strings that don't contain c, and both start and end with a.

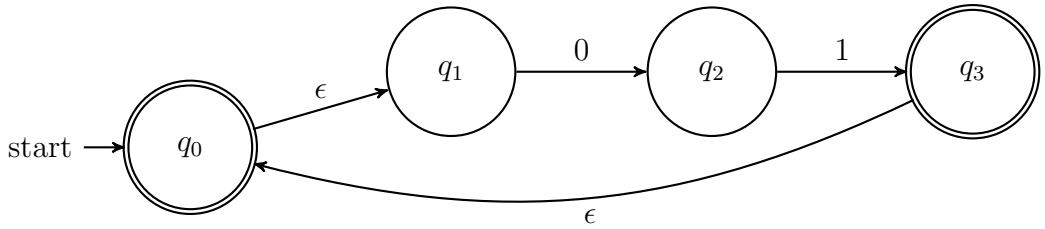
2.



Solution.

The strings that contain at least one 1 and the number of 0's after the last 1 is even.

3.



Solution.

The strings that are a repetition of 01 (including empty string, i.e., repeating zero times).

If we were to turn the NFA given above into an equivalent DFA (whose states now correspond to subsets of $\{q_0, q_1, q_2, q_3\}$), what state would be the start state? Which states would be accept states? What's the transition for state $\{q_1, q_2\}$ when reading a 1?

Solution.

The start state is $\{q_0, q_1\}$, which is the ϵ-closure of q_0. The accept states are all subsets that contain either q_0 or q_3. Among the states reachable from the start state, these are $\{q_0, q_1\}$ and $\{q_0, q_1, q_3\}$.
After reading a 1 from the state $\{q_1, q_2\}$, the NFA transitions to the state $\{q_0, q_1, q_3\}$. (We first move from q_2 to q_3 on input 1, and then take the ϵ-closure.)

RegEx

Construct regular expressions for the following languages.

1. $L_1 = \{w \in \{a, b, c\}^* \mid w$ does contain the substring $abc\}$

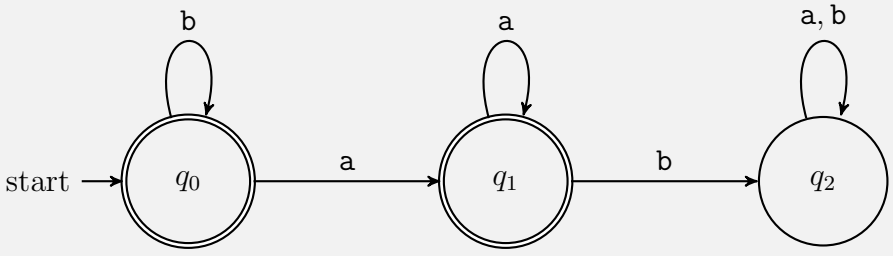
   Solution.

   $(\mathtt{a} \cup \mathtt{b} \cup \mathtt{c})^*\mathtt{abc}(\mathtt{a} \cup \mathtt{b} \cup \mathtt{c})^*$.

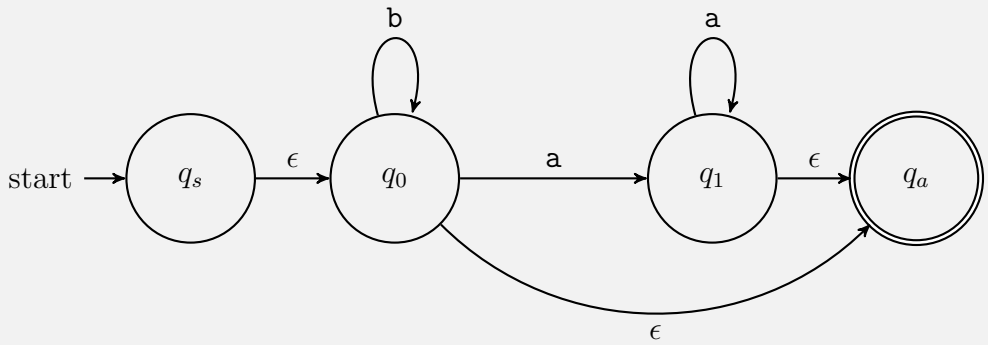
2. $L_2 = \{w \in \{a, b\}^* \mid$ does not contain the substring $ab\}$.

   Solution.

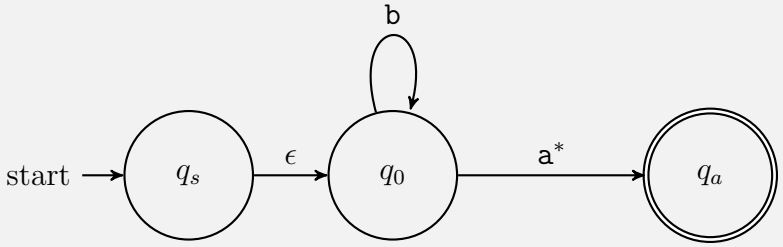
   We first construct a DFA that decides L_2:



   Then we convert the DFA to a regular expression using the standard procedure. First, we create an NFA where we add a new start state q_s and a new accept state q_a (state q_2 can be omitted):



Contracting state q_1 (note that $\epsilon \cup \mathtt{aa}^* = \mathtt{a}^*$), we get:



Finally we contract state q_0 and get the regular expression $\mathtt{b}^*\mathtt{a}^*$.
An alternative solution is to notice that the condition "does not contain the substring ab" is equivalent to "does not contain the subsequence ab": Clearly the latter implies the former. For the converse, suppose a string w contains ab as a subsequence, say $w_i = \mathtt{a}$ and $w_j = \mathtt{b}$ for some $i < j$. Then let $t > i$ be the first position that $w_t \neq w_{t-1}$, then $w_t = \mathtt{b}$ and $w_{t-1} = \mathtt{a}$, so w contains ab as a substring.
Thus, L_2 is the language of strings that do not contain ab as a subsequence. This is equivalent to all a's appearing after all b's, which is exactly the language described by the regular expression $\mathtt{b}^*\mathtt{a}^*$.

Prove or disprove the following equivalences between regular expressions. Assume r, s, and t are regular expressions.

1. $s^*(rs^*r)^*s^* = (r \cup s)^*$

Solution.

Not equivalent.

Proof. For example take $r = \mathtt{a}$ and $s = \mathtt{b}$.
Any string accepted by regular expression on the left must contain an even number of a's: every $\mathtt{b}^*$ has zero a's, and every $(\mathtt{ab}^*\mathtt{a})$ has two a's so the total number of a's is even.
So the string $w = \mathtt{a}$ is accepted by the regular expression on the right but not by the regular expression on the left. □

2. $r(s \cup t)^* = (rs)^* \cup (rt)^*$

Solution.

Not equivalent.

Proof. Specialize to $s = t = \epsilon$, then the left hand side is $r\epsilon^* = r$, and the right hand side is $(r\epsilon)^* \cup (r\epsilon)^* = r^*$. Further take $r = \mathtt{a}$, then the left hand side accepts only the string $\mathtt{a}$, while the right hand side is $\mathtt{a}^*$, clearly not equivalent. □

3. $(r^* \cup s^*)^* = (r \cup s)^*$

Solution.

Equivalent.

Proof. If a string w is accepted by the regular expression on the left, then w can be written as $w = w_1 w_2 \cdots w_k$ where each w_i is accepted by $r^* \cup s^*$, i.e. each w_i is accepted by either r^* or s^*. If w_i is accepted by r^*, then w_i can be written as $w_i = w_{i1} w_{i2} \cdots w_{i\ell_i}$ where each w_{ij} is accepted by r. Similarly if w_i is accepted by s^*, then w_i can be written as $w_i = w_{i1} w_{i2} \cdots w_{i\ell_i}$ where each w_{ij} is accepted by s. In either case, write w as

$$w = w_{11} w_{12} \cdots w_{1\ell_1} w_{21} w_{22} \cdots w_{2\ell_2} \cdots w_{k1} w_{k2} \cdots w_{k\ell_k},$$

where each w_{ij} is accepted by either r or s. Thus, w is accepted by the regular expression on the right.
Conversely, if a string w is accepted by the regular expression on the right, then w can be written as $w = w_1 w_2 \cdots w_k$ where each w_i is accepted by either r or s. If w_i is accepted by r, then w_i is also accepted by r^*; if w_i is accepted by s, then w_i is also accepted by s^*. Thus, in either case, w_i is accepted by $r^* \cup s^*$. Therefore, w is accepted by the regular expression on the left. □

4. $r^*(s \cup t)^* = (r \cup s \cup t)^*$

Solution.

Not equivalent.

Proof. For example take $r = \mathtt{a}$, $s = \mathtt{b}$ and $t = \epsilon$. The two sides become $\mathtt{a}^*\mathtt{b}^*$ and $(\mathtt{a} \cup \mathtt{b})^*$ respectively. The string $w = \mathtt{ba}$ is accepted by RHS but not by LHS. □

Asymptotic Notation

For each of the following functions, circle *all* of the letters that are true.

1. $f(n) = (\log_2 n)^2$

   (a) $f(n) = O(\log_{10} n)$
   (b) $f(n) = O((\log_{10} n)^2)$
   (c) $f(n) = O(n^{1/10})$

Solution.

(b) (c).

2. $f(n) = 2^n$

   (a) $f(n) = \Omega(2^{\sqrt{n}})$

   (b) $f(n) = \Omega(n^2)$

   (c) $f(n) = \Omega((\log_2 n)^n)$

Solution.

(a) (b).

Streaming

1. Give a streaming algorithm for recognizing the language

$$L = \{w \in \{0,1,2\}^* \mid \text{the number of 2's in } w \text{ is at least } 2\times(\text{number of 1's in } w)\}$$

(For example, $00121222 \in L$ because there are four 2's and two 1's, and $4 \geq 2 \times 2$. $112022 \notin L$ because there are three 2's and two 1's, and $3 \ngeq 2 \times 2$.)

Also, prove that your streaming algorithm uses $O(\log n)$ space.

Solution.

Consider the streaming algorithm $\mathcal{A}$ with

- Alphabet set $\Pi = \Sigma = \{0,1,2\}$.
- Initialize with $I = 020$.
- Update rule δ: For any $w \in \Pi^*$, if w has the form $x_1 2 x_2$ where $x_1, x_2 \in \{0,1\}^*$, then treat x_1, x_2 as binary numbers and let

$$\delta(w,\sigma) = \begin{cases} x_1 2 x_2 & \text{if } \sigma = 0, \\ (x_1+1)2x_2 & \text{if } \sigma = 1, \\ x_1 2(x_2+1) & \text{if } \sigma = 2. \end{cases}$$

  Otherwise let $\delta(w,\sigma) = w$.

- Accepting rule $\gamma(w) = 1$ iff w has the form $x_1 2 x_2$ where $x_1, x_2 \in \{0,1\}^*$, and $x_2 \geq 2 \cdot x_1$ as binary numbers.

By mathematical induction on the length of the input string w, we can show that after reading w, the algorithm $\mathcal{A}$ is in state $x_1 2 x_2$ where x_1 is the number of 1's in w and x_2 is the number of 2's in w. Thus, w is accepted by $\mathcal{A}$ if and only if $w \in L$. Since x_1, x_2 are both at most n, their length is at most $\log_2(n+1)+1$. Therefore, the space used by $\mathcal{A}$ is at most $2(\log_2(n+1)+1)+1 = O(\log n)$.

2. Give a streaming algorithm for recognizing the language

$$L = \{w \in \{0,1,2\}^* \mid \text{the most common symbol in } w \text{ appears } k \text{ times and the least common appears } k-1 \text{ times}\}.$$

Ex: $w = 001112 \notin L$ since the most common element is 1 and it appears 3 times and the least common element 2 appears only 1 time. But $w = 0011122$ is in L (most common appears 3 times, least common appears 2 times).

Also, prove that your streaming algorithm uses $O(\log n)$ space.

Solution.

Consider the streaming algorithm $\mathcal{A}$ with

- Alphabet set $\Pi = \Sigma = \{0,1,2\}$.
- Initialize with $I = 02020$.
- Update rule δ: For any $w \in \Pi^*$, if w has the form $x_0 2 x_1 2 x_2$, where $x_0, x_1, x_2 \in \{0,1\}^*$, then treat x_0, x_1, x_2 as binary numbers and let

$$\delta(w,\sigma) = \begin{cases} (x_0+1)2x_1 2x_2 & \text{if } \sigma = 0, \\ x_0 2(x_1+1)2x_2 & \text{if } \sigma = 1, \\ x_0 2x_1 2(x_2+1) & \text{if } \sigma = 2. \end{cases}$$

  Otherwise let $\delta(w,\sigma) = w$.
- Accepting rule $\gamma(w) = 1$ iff w has the form $x_0 2 x_1 2 x_2$ where $x_0, x_1, x_2 \in \{0,1\}^*$, and $\max\{x_0, x_1, x_2\} = \min\{x_0, x_1, x_2\} + 1$ as binary numbers.

By mathematical induction on the length of the input string w, we can show that after reading w, the algorithm $\mathcal{A}$ is in state $x_0 2 x_1 2 x_2$ where x_i is the number of i's in w for each $i = 0, 1, 2$. Thus, w is accepted by $\mathcal{A}$ if and only if $w \in L$. Since x_0, x_1, x_2 are all at most n, their length is at most $\log_2(n+1)+1$. Therefore, the space used by $\mathcal{A}$ is at most $3(\log_2(n+1)+1)+2 = O(\log n)$.

Proving Non-Regularity

Show that the following languages are *not* regular. Do this by giving a lower bound on the one-way communication complexity of language, or by using a direct argument. You should practice proving the lower bound by both a direct argument, and by a one-way communication complexity lower bound argument. Do not prove your lower bound by a reduction from another language.

1. $L = \{w \in \{0,1\}^* \mid w = xy$ and the first 1 in x does not appear after the first 1 in $y\}$.

   Note: this is the example "First" given in the class notes, so you already have the answer. Practice solving it again without looking.

Solution.

See class notes.

2. Index $= \{w \in \{0,1\}^* \mid w = xy \text{ and } x_i = 1 \text{ where } i \text{ is the first index in } y \text{ with } y_i = 1\}$.

   Note: this is an example from the lecture notes. But we didn't cover the lower bound in class.

Solution.

This problem has a one-way communication lower bound of $\Omega(n)$.

Proof. Suppose it uses $< n$ bits to decide Index where $|x| = |y| = n$. Then the possible messages that Alice can send to Bob are the strings of length $< n$, which are at most $2^n - 1$ many. But there are 2^n possible inputs for Alice, so by the pigeonhole principle, there exist two distinct inputs x and x' for Alice that lead to the same message m being sent to Bob. Since $x \neq x'$, there is some index i such that $x_i \neq x'_i$. Without loss of generality, assume $x_i = 0$ and $x'_i = 1$.
Now let Bob's input be $y = 0^{i-1}10^{n-i}$, i.e., the string of n bits with a 1 in the i-th position and 0's elsewhere. Then the first index in y with $y_i = 1$ is i, so Bob should accept if Alice's input is x' (since $x'_i = 1$), and reject if Alice's input is x (since $x_i = 0$). But Bob receives the same message m from Alice in both cases, so he must either accept both or reject both, a contradiction. Therefore, the one-way communication complexity of Index is at least n bits. □

3. Practice proving one-way communication lower bounds for other examples that were proven in class or in the notes (for example: EQ, MAJ, and variants of MAJ such as the language Sum from HW3.)

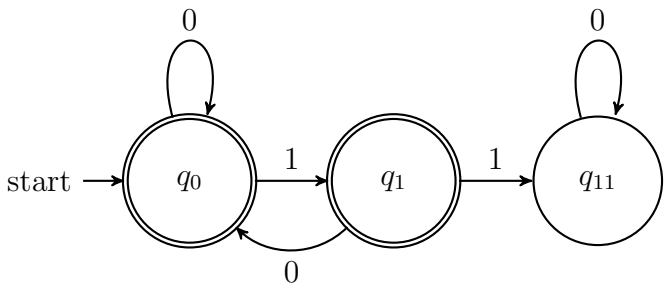
Solution.

See this note and this note.

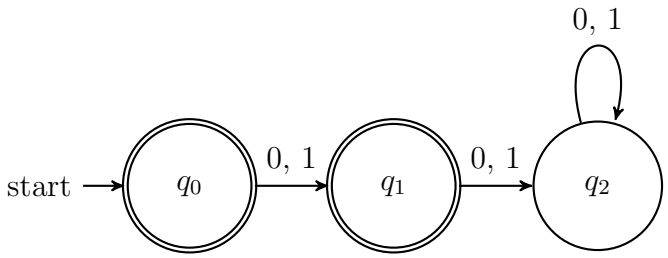
True/False Section

Determine whether each of the following statements is true or false:

1. (T/**F**) The following is a valid DFA for a language over $\{0,1\}$:



2. (T/**F**) The language recognized by the following DFA is $\{0,1\}$:



3. (**T**/F) Suppose the languages L_1 and L_2 are recognized by the DFAs D_1 and D_2, respectively. If D_1 has 3 states and D_2 has 4 states, then there exists a 15 state DFA that recognizes $L_1 \backslash L_2 = \{w \in L_1 \text{ and } w \notin L_2\}$.

4. (T/**F**) If L_1 and L_2 are languages with $(L_1)^* = L_2$, and L_2 is regular, then L_1 is regular.

5. (T/**F**) If L_1 and L_2 are non-regular languages, then $L_1 \cup L_2$ is a non-regular language as well.

6. (T/**F**) The concatenation of a language with itself, $L^2 = LL$, can be written in set-theoretic notation as $\{ww \mid w \in L\}$.

7. (T/**F**) Suppose there exists an NFA $N = (Q, \Sigma, \delta, q_0, F)$ which recognizes the language L. Let $N' = (Q, \Sigma, \delta, q_0, Q \backslash F)$ be the NFA where the accept and reject states of N are switched. Then N' recognizes the complement $\overline{L}$.

8. (T/**F**) If L is a regular language recognized by an NFA with $k \in \mathbb{Z}_{>0}$ states, then there exists an DFA recognizing L with $2k - 1$ states.

9. (**T**/F) If the language L is recognized by an NFA with 4 states, then there exists a DFA recognizing L with less than 20 states.

10. (**T**/F) The language $L = \{w \in \{0,1\}^* \mid w \text{ has even length and contains the substring } 1010\}$ can be described with a regular expression.

11. (T/**F**) The language described by the expression $(01)^*(0 \cup 1)^* \varnothing$ contains more strings than the language described by the expression $((0 \cup 10) \cup \epsilon)(0 \cup 1)110$.

12. Quick asymptotic notation:

    (a) (**T**/F) $n \log^2 n = O(n^2)$
    (b) (**T**/F) $n = \Omega(\log n)$
    (c) (T/**F**) $1/n = \Omega(1/\log n)$
    (d) (**T**/F) $2^n = o(3^n)$
    (e) (**T**/F) $3^n = 2^{O(n)}$
    (f) (**T**/F) $n^{-1} = \text{poly}(n)$

13. More challenging asymptotic notation:

    (a) (T/**F**) $\sum_{i=1}^{n} 3^i = O(2^n)$
    (b) (T/**F**) $n! = \text{poly}(n)$