Lecture 18:

- HW4 due Thursday at 11 am
  - $\hookrightarrow$ only 1 late HW !!!

- See the exercises solutions

- For the proofs of decidable / recognizable / undecidable

(1) If input $\in L$, then accept

(2) If input $\notin L$, then doesn't accept / reject

Today: chapter 4.2 + 5.1

Goal: A_{TM} is not decidable

Idea: There are more languages than TMs

A simple question:

$\mathbb{N} = \{1, 2, 3, 4 \ldots\}$ $\quad\quad$ $S = \{2, 4, 6, 8, 10 \ldots\}$

$\mathbb{N}$

do these 2 sets have the same size?

Defn: A set S is countable if there exists an injective map from S to $\mathbb{N}$.

Injective : $f(x) = f(y) \Rightarrow x = y$

never have two different elements mapped to the same thing.

(1) Finite sets are countable

Ex: $S = \{3, 10, 7, 4\}$

$f: S \to 1$

$$\begin{array}{cccc} 3, & 10, & 7, & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \end{array}$$

Idea: a set S is countable if you can make a list where every element in S appears exactly once.

(2) Examples of infinite countable sets

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

$$0, 1, -1, 2, -2, 3, -3, \dots$$

A countable set is either finite or it has the same size as $\mathbb{N}$

Other example: all pairs $(i,j) \in \mathbb{N} \times \mathbb{N}$

	1	2	3
1	(1,1)	(1,2)	(1,3)
2	(2,1)	(2,2)	
3	(3,1)		

$(1,1)$, $(1,2)$, $(2,1)$,

$(1,3)$ ....

$\mathbb{Q} = \{ \frac{n}{m} \mid n, m \in \mathbb{N} \}$, also countable.

Assume $\Sigma = \{0, 1\}$

Idea: set of all TMs is countable.

Foreach TM, can represent it as a string $\langle M \rangle$

We can order all $\langle M \rangle$ in lexicographic order

$\hookrightarrow \varepsilon, 0, 1, 00, 01, 10, 11 \ldots$

get an enumeration $\langle M_1 \rangle, \langle M_2 \rangle, \langle M_3 \rangle, \ldots$

$\hookrightarrow$ every TM has its description in this list.

$\hookrightarrow$ can then order all TMs as

$M_1, M_2, M_3 \ldots$

The set of all languages over $\{0,1\}$ is NOT countable.

More languages than TMs. So $\exists$ some language L, such that no TM has $L(M) = L$

↳ there are some languages which are not recognizable.

> to prove this: use DIAGONALIZATION

Cantor: showed $\mathbb{R}$ is not countable

↳ real numbers

We will show that

$D = \{ \langle M \rangle \mid M \text{ is a TM and it doesn't accept } \langle M \rangle \}$

diagonal language.

is not recognizable

W.T.S.: no TM M^* has $L(M^*) = D$

Proof:

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	M^*
M_1	1	0	1 ← 1 if M_1 accepts $\langle M_3 \rangle$	

0 otherwise

M_2	1	0	1
M_3	0	0	1
M^*			?

← M_3 accepts $\langle M_3 \rangle$ so $\langle M_3 \rangle \notin D$

$\langle M_2 \rangle \in D$ because M_2 doesn't accept $\langle M_2 \rangle$

Assume for contradiction $L(M^*) = D$

$D = \{ \langle M \rangle \mid M \text{ is a TM and it doesn't accept } \langle M \rangle \}$

(1) If M^* accepts $\langle M^* \rangle$ then, $\langle M^* \rangle \notin D$

so M^* should not accept $\langle M^* \rangle$

Since $L(M^*) = D$, M^* should not accept $\langle M^* \rangle \notin D$

(2) If M^* doesn't accept $\langle M^* \rangle$

$\langle M^* \rangle \in D$ (by definition of D)

since $L(M^*) = D$, M^* accepts $\langle M^* \rangle$

contradiction

No matter what M^* does on $\langle M^* \rangle$ we get a contradiction. So there doesn't exist a TM M^* with $L(M^*) = D$

$\hookrightarrow$ D is not recognizable!

Back to A_{TM}

$D = \{ \langle M \rangle \mid M$ is a TM and it doesn't accept $\langle M \rangle \}$

$\tilde{D} = \{ \langle M \rangle \mid M$ is a TM & it accepts $\langle M \rangle \}$

$\overline{D} = \tilde{D} \cup \{ w \mid w$ isn't of the form $\langle M \rangle \}$

$\tilde{D}$ is decidable iff $\overline{D}$ is recognizable

Punchline: D is not recognizable

$\Rightarrow \overline{D}$ is not decidable

$\Rightarrow \tilde{D}$ is not decidable

If $\overline{D}$ is decidable, D is decidable

(1) $\tilde{D}$ is recognizable

Now we show $\tilde{D} \leq_T A_{TM}$, since $\tilde{D}$ is not decidable A_{TM} is not decidable.

Assume we have a decider N for A_{TM}, we build a decider M' for $\tilde{D}$.

$\tilde{D} = \{ \langle M \rangle \mid M \text{ is a TM \& it accepts } \langle M \rangle \}$

Proof of correctness:

- clearly M' always halts

(1) If $\langle M \rangle \in \tilde{D}$, then M accepts $\langle M \rangle$

$\langle M, \langle M \rangle \rangle$ is in A_{TM}, so N accepts

on line (1), so M' accepts.

(2) If $\langle M \rangle \notin \tilde{D}$, then M does not accept

$\langle M \rangle$, so $\langle M, \langle M \rangle \rangle \notin A_{TM}$, so N

rejects on line (1), so M' rejects.

So M' decides $\tilde{D}$, so $\tilde{D} \leq_T A_{TM}$.

$L = \{ \langle M \rangle \mid M$ accepts every string starting with $11 \}$

We show $A_{TM} \leq_T L$, so L is not decidable because A_{TM} is not decidable

Let N be a decider for L, we build decider D for A_{TM}.

D: on input $\langle M, w \rangle$

(1) Let M' be the following TM:

(a) On input x:

(b) If x doesn't start with 11 : reject

(c) Else:

(d) run M on w ← stuck here if M loops on w

(e) if M accepts : accept x

(f) if M rejects : reject x

(2) Run N on $\langle M' \rangle$ ← accept or reject since N is a decider

(3) If N accepted : accept

(4) If N rejected : reject

We now prove this is a decider for A_{TM}, clearly D always halts.

(1) M' rejects every string that does not start with 11

(2) Let x start with 11, we have 3 cases:

M loops on w: M' loops on x on line (d)

M rejects w : M' rejects x on line (f)

M accepts w: M' accepts x on line (e)

So: If M accepts w: $L(M') = \{x \mid x \text{ starts with } 11\}$

So $\langle M' \rangle \in L$

If M doesn't accept w: $L(M') = \emptyset$

So $\langle M' \rangle \notin L$

Hence we have

(1) If $\langle M, w\rangle \in A_{TM}$, then M accepts w

So $\langle M'\rangle \in L$.

so N accepts on line 2

So D accepts $\langle M, w\rangle$

(2) If $\langle M, w\rangle \notin A_{TM}$, then M doesn't accept w, so $\langle M'\rangle \notin L$

so N rejects on line 2

So D rejects $\langle M, w \rangle$

So D decides A_{TM}, so $A_{TM} \leq_T L$

as claimed.