Lecture 17:

Hw 4 is out: please write <u>proofs</u>

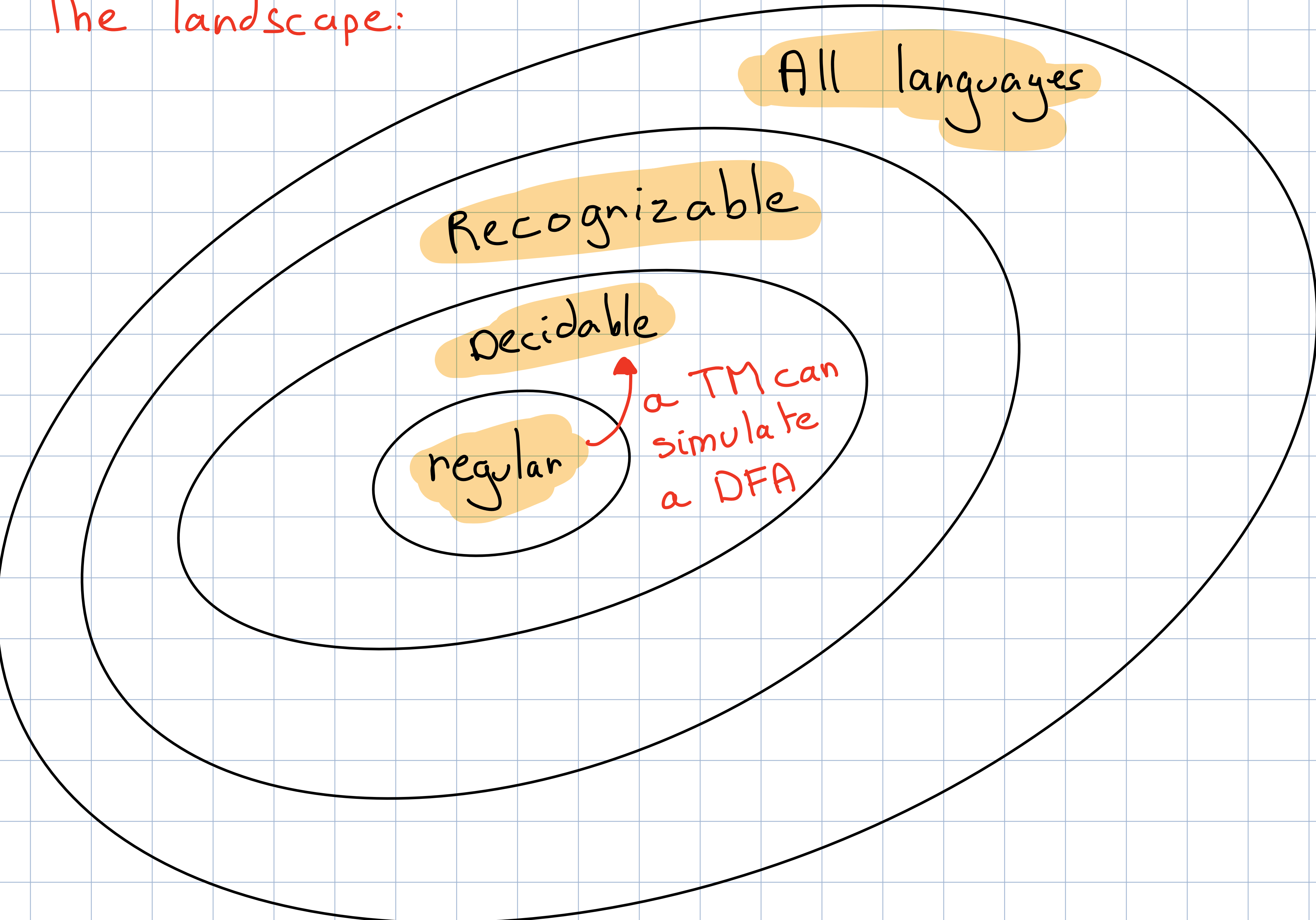
↳ due the 13 th

look at chapter 4 + 5 for a lot of examples + today

Tip: do the exercises + lots of practice

The proofs are "always the same"
it just takes some intuition building

The landscape:



1. If L is decidable $\Rightarrow$ L is recognizable

2. Closure under $\cup$, $\cap$, $\overline{}$ for decidable

   If L_1, L_2 are decidable:

   $L_1 \cap L_2$, $L_1 \cup L_2$, $\overline{L_1}$, $\overline{L_2}$ are decidable

not closed under complement.

3. Closure under $\cap$, $\cup$ for recognizable

   If L_1, L_2 are recognizable:

   $L_1 \cap L_2$, $L_1 \cup L_2$ are recognizable

Today chapter 5.1 & 5.3

4.22

Thm: If $L, \bar{L}$ are recognizable, L is decidable

Proof: Build a TM M with $L(M) = L$, and M always halts.

Let M_1, M_2 be TM recognizing L and $\bar{L}$

issue: M_1, M_2 might loop on an input w.

M:

on input w

(1) For $i = 1, 2, \ldots$

i transitions

Dove failing

(2) Run M_1 on w for i steps

(3) If M_1 accepted $\Rightarrow$ accept w

$\hookrightarrow w \in L$

(4) Run M_2 on w for i steps

(5) If M_2 accepted $\Rightarrow$ reject w

$\hookrightarrow w \in \bar{L}$

Claim: M always halts + accepts iff $w \in L$

$\forall w$: exactly one of $M_1(w)$ or $M_2(w)$ halts and accepts

$\therefore \forall w$ there is some time step i such that either (i) $M_1(x)$ halts + accepts in i steps

(ii) $M_2(x)$ halts + accepts in i steps

If (i) then $x \in L$ and $M(x)$ accepts on line (3)

If (ii) then $x \notin L$ and $M(x)$ rejects on line (5)

once i in the for loop is large enough

Recall $A_{TM} = \{\langle M, x\rangle \mid M$ is a TM and M accepts $x\}$

- We saw A_{TM} is recognizable

Thm: A_{TM} is not decidable $\Rightarrow$ proof next time

Corollary: $\overline{A_{TM}}$ is not recognizable

Proof: If $\overline{A_{TM}}$ was recognizable then A_{TM} would be decidable by closure property

Small note $\overline{A_{TM}} \neq \overbrace{\{\langle M, x\rangle \mid M \text{ is a TM and it doesn't accept } x\}}^{\tilde{A}_{TM}}$

$\to$ this also contains all strings w which are NOT of the form $\langle M, x\rangle$

$\overline{A_{TM}}$ is decidable iff $\tilde{A}_{TM}$ is

6.3

Reductions : Turing Reductions

Defn: Language A is Turing reducible to language B, written $A \leq_T B$ if: a decider for B implies that there exists a decider for A

Key point: If $A \leq_T B$ and B is decidable then A is decidable

Corallary: If $A \leq_T B$ and A is not decidable, then B is not decidable

Example: $\text{Halt}_{TM} = \{\langle M, w\rangle \mid M$ is a TM and it halts on $w\}$

Lemma 1: Halt is recognizable

↳ exercise

Lemma 2: Halt is not decidable

Proof: we show $A_{TM} \leq Halt_{TM}$

Since A_{TM} is undecidable $\Rightarrow$

$Halt_{TM}$ is undecidable

Assume there exists a decider N for Halt, we buid a decider for A_{TM}.

reject inputs with bad format

D : On input $\langle M, x \rangle$

1) Run N on $\langle M, x \rangle$

↳ this accepts / rejects

2) If N rejects $\Rightarrow$ halt + reject

↳ M loops on x

$\Rightarrow$ M halts on x

3) If N accepts:

4) Run M on x) this always halts

5) If M accepts $\Rightarrow$ accept

6) If M rejects $\Rightarrow$ reject

Proof of correctness: N always halts on line 1

If M loops on x => N rejects =>
we reject + halt on line

If M doesn't loop on x => N accepts
& we go to line 4. On line 4: M always
halts. On line 5/6 we output accept or
reject: we output accept iff M accepted
x.

So D always halts + accepts iff M accepts x.

So D is a decider for A_{TM}. So $A_{TM} \leq_T Halt$

Corallary: $\overline{Halt}$ is not recognizable

If Halt & $\overline{Halt}$ were both recognizable, Halt would be decidable ↯.

$$E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \}$$

① $\overline{E_{TM}}$ is recognizable: dove tailing

Sketch: give a recognizer R for $\overline{E_{TM}}$

R: on input $\langle M \rangle$: ↪ auto accept inputs that are not of the form $\langle M \rangle$

(1) For $i = 1, 2, \dots$

(2) Run M on all strings w with $|w| \leq i$ for i steps.

(3) Accept if M accepts any strings

Proof: assume $\langle M \rangle \notin \overline{E_{TM}}$. M must accept at least one string x^*.

Let $l = |x^*|$, t the number of time steps $M(x^*)$ takes to accept

When $i \geq$ max (l, t) we run M

line 2

on x^* for t steps and accept
line 3

assuming we haven't accepted before

assume $\langle M \rangle \in E_{TM}$: $L(M) = \emptyset \Rightarrow$ clearly R can't accept since R accepts iff we found a string M accepts.

R accepts iff $\langle M \rangle \in \overline{E_{TM}}$ so $\overline{E_{TM}}$ is recognizable

② E_{TM} is not decidable (E_{TM} is not recognizable)

Show $A_{TM} \leq_T E_{TM}$: A_{TM} is undecidable

so E_{TM} is undecidable

Let N be a decider for E_{TM}, we build a decider D for A_{TM}:

D: on input $\langle M, x\rangle$

(1) Let M' be the following TM

M' can depend on $\langle M, x\rangle$

M': on input y

- If $y \neq x$: reject

$y = x \rightarrow$ - Else run M on x &

$y \notin L(M')$ if $y \neq x$

$x \in L(M')$ iff M accepts

output what M outputs) x

D can write
$\langle M' \rangle$

If M doesn't accept x
$L(M') = \emptyset$

(2) Run N on $\langle M' \rangle$

(3) If N rejects => accept

(4) If N accepts => reject

Proof: M' is as follows: $L(M') = \emptyset$ if M doesn't accept x

$L(M') = \{x\}$ if M accepts x

Since N is a decider it accepts/rejects on line 2 + D always halts

(1) If $\langle M, x \rangle \in A_{TM} \Rightarrow$ M accepts x so

$L(M') \neq \emptyset$ so N rejects and D accepts

(2) If $\langle M, x \rangle \notin A_{TM} \Rightarrow$ M doesn't accept x so

$L(M') = \emptyset$ so N accepts and D rejects

D always halts + accepts iff $\langle M, x \rangle \in A_{TM}$

So D is a decider for A_{TM}. So $A_{TM} \leq_T E_{TM}$