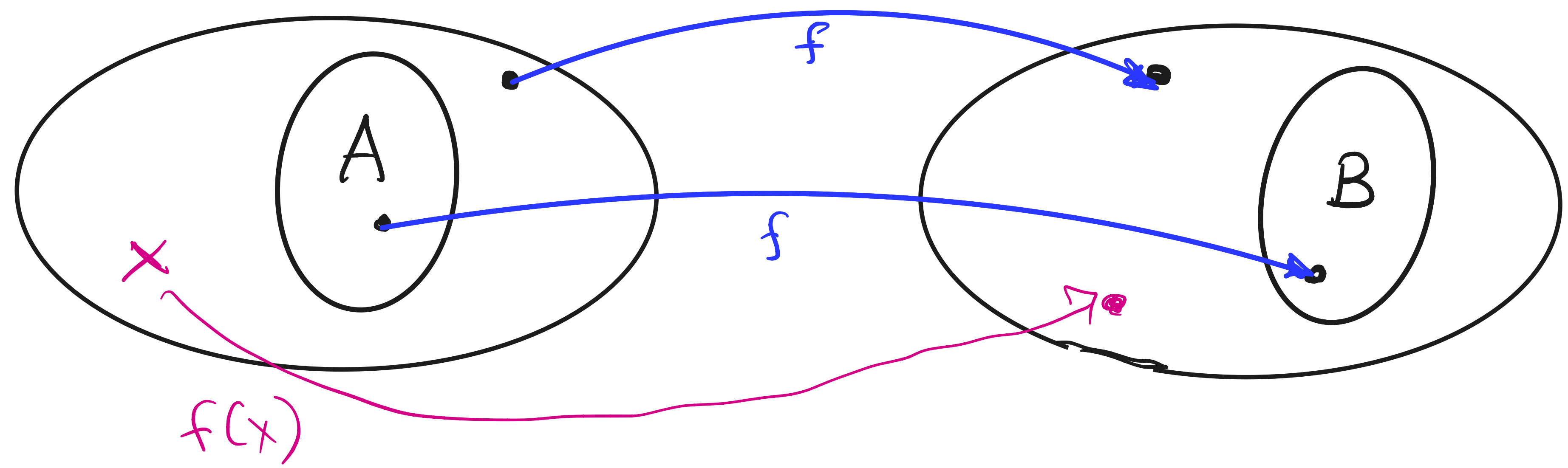
Lecture 21

Today : NP, NP-completeness

NP - completeness

Definition Language A is polynomial-time (mapping) reducible to B (written $A \leq_p B$) if there is a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ such that $w \in A \iff f(w) \in B$



Definition

- A language $B \subseteq \{0,1\}^*$ is NP-hard if for <u>every</u> $A \in NP$ there is a polynomial time reduction from A to B $(A \leq_p B)$

NP - completeness

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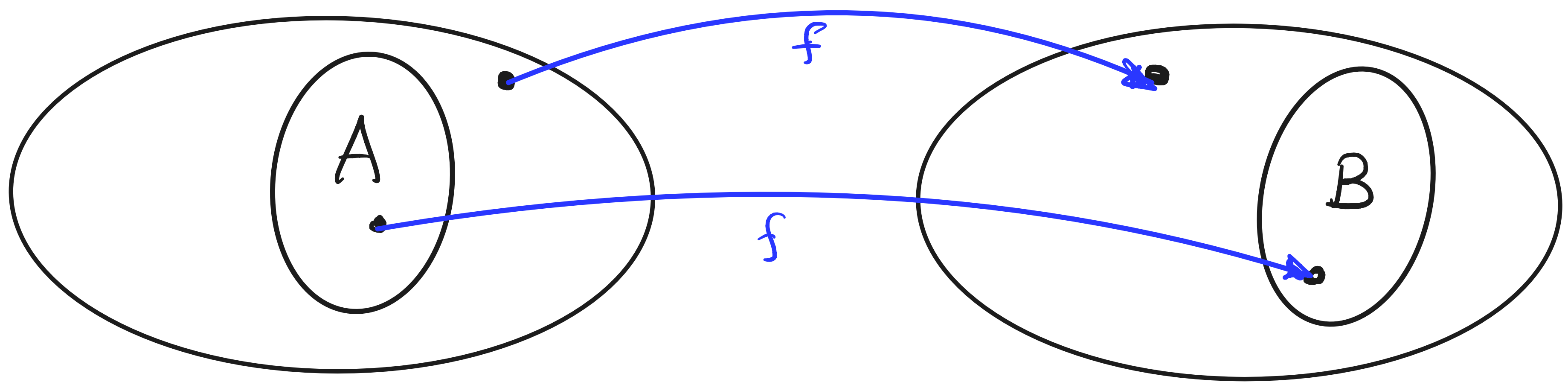


Definition

- A language $B \subseteq \{0,1\}^*$ is NP-hard if for <u>every</u> $A \in NP$ there is a polynomial time reduction from A to B ($A \leq_p B$)

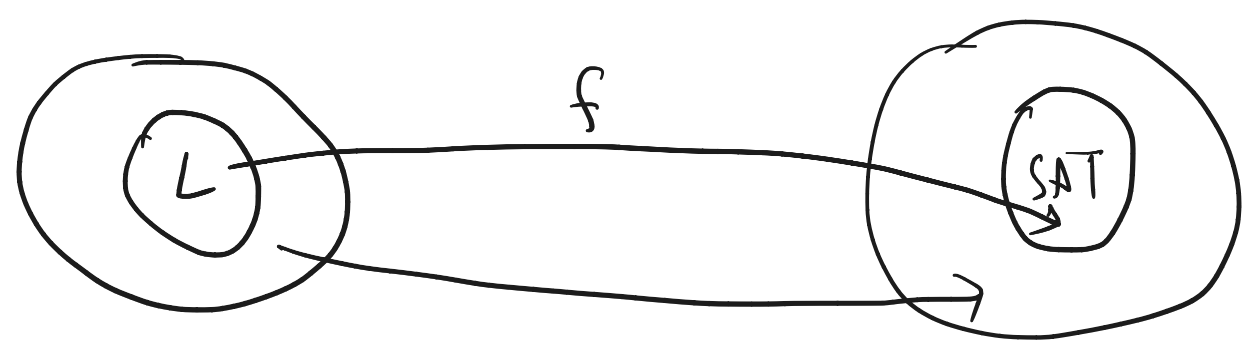
- $B \subseteq \{0,1\}^*$ is NP-complete if: (i) B is in NP and
  (ii) B is NP-hard

NP - completeness

Cook - Levin theorem

SAT and 3-SAT are NP-complete (Proof Next week)

1. SAT, 3SAT in NP
2. SAT, 3SAT are NP-hard : meaning for every
   Language $L \in NP$, we can solve L in poly time ($L \in P$)
   iff SAT (3SAT) $\in P$!



To solve L : on x, compute $f(x)$; Run SAT on input $f(x)$

NP - completeness

Cook - Levin theorem

For every $k \geq 3$ 3-SAT is NP-complete

(Proof Next week)

To show another language L is NP complete we just need to show:

(1) $L \in NP$

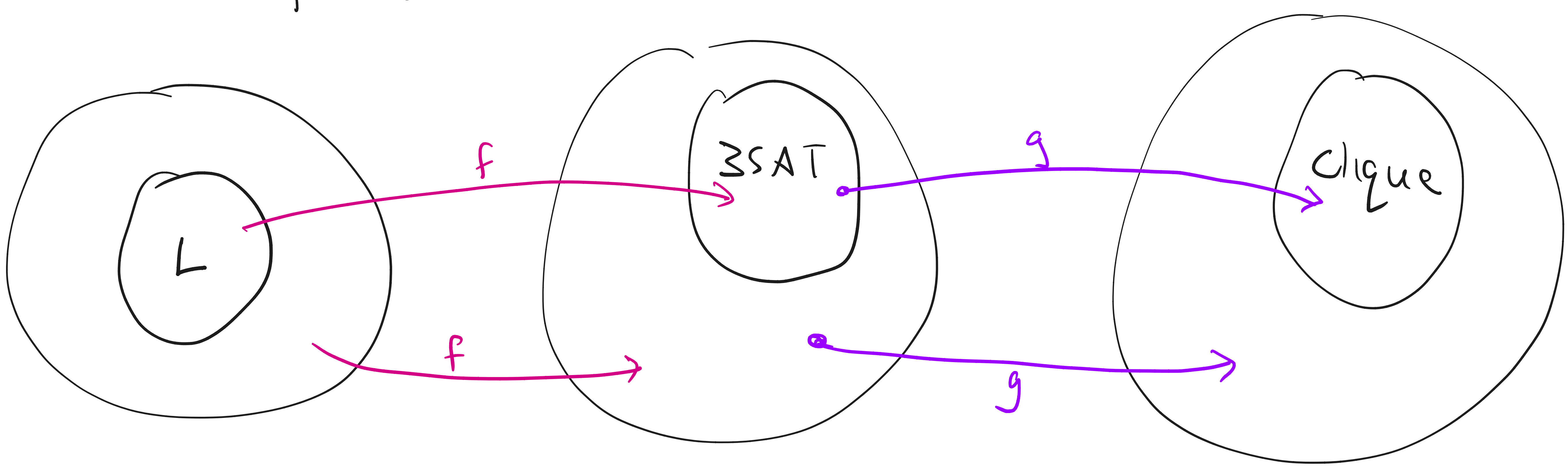
(2) Show $3SAT \leq_p L$

NP - completeness

Today: Assume SAT, 3SAT are NP-complete.
Leverage this to show some other languages in NP are also NP-complete

Say we want to prove CLIQUE NP-complete

1. Clique $\in NP$
2. 3SAT $\leq_p$ clique



NP-completeness via Reductions

Clique:
Input : $(G = V, E)$, K
accept iff G contains a clique of size K
(clique is a fully connected subgraph of G of size k)

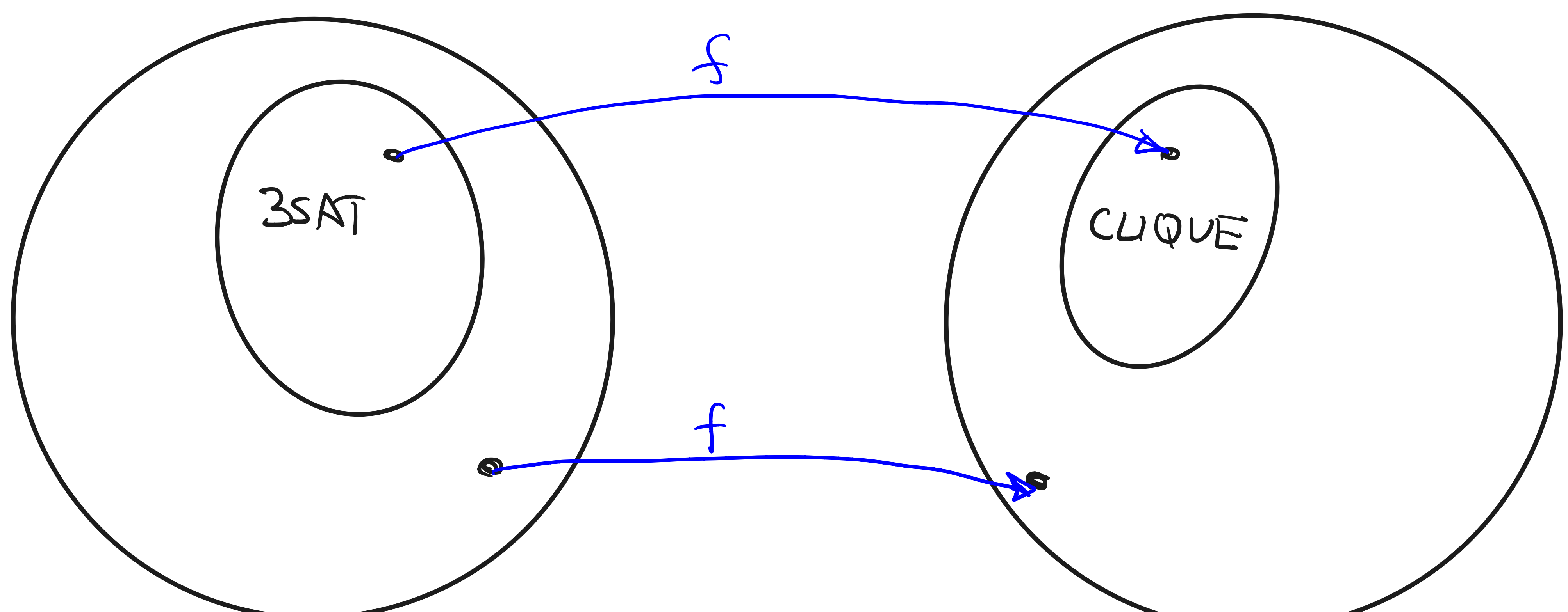
① Clique

<u>Theorem</u> CLIQUE is NP-complete

<u>Proof</u> (2 steps)

(i) CLIQUE is IN NP (we already showed this)

(ii) CLIQUE is NP-HARD: Need to show $3SAT \leq_p CLIQUE$



3-SAT $\le_p$ CLIQUE

Example

Let $\phi = \underbrace{(x_1 \vee x_2)}_{C_1} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee x_3)}_{C_2} \wedge \underbrace{(x_1 \vee \bar{x}_3 \vee x_4)}_{C_3} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_4)}_{C_4}$

m clauses
here $m = 4$

$f : \phi \longrightarrow (G_\phi, k = m)$

G_ϕ: $|V| = |C_1| + |C_2| + \dots + |C_m|$

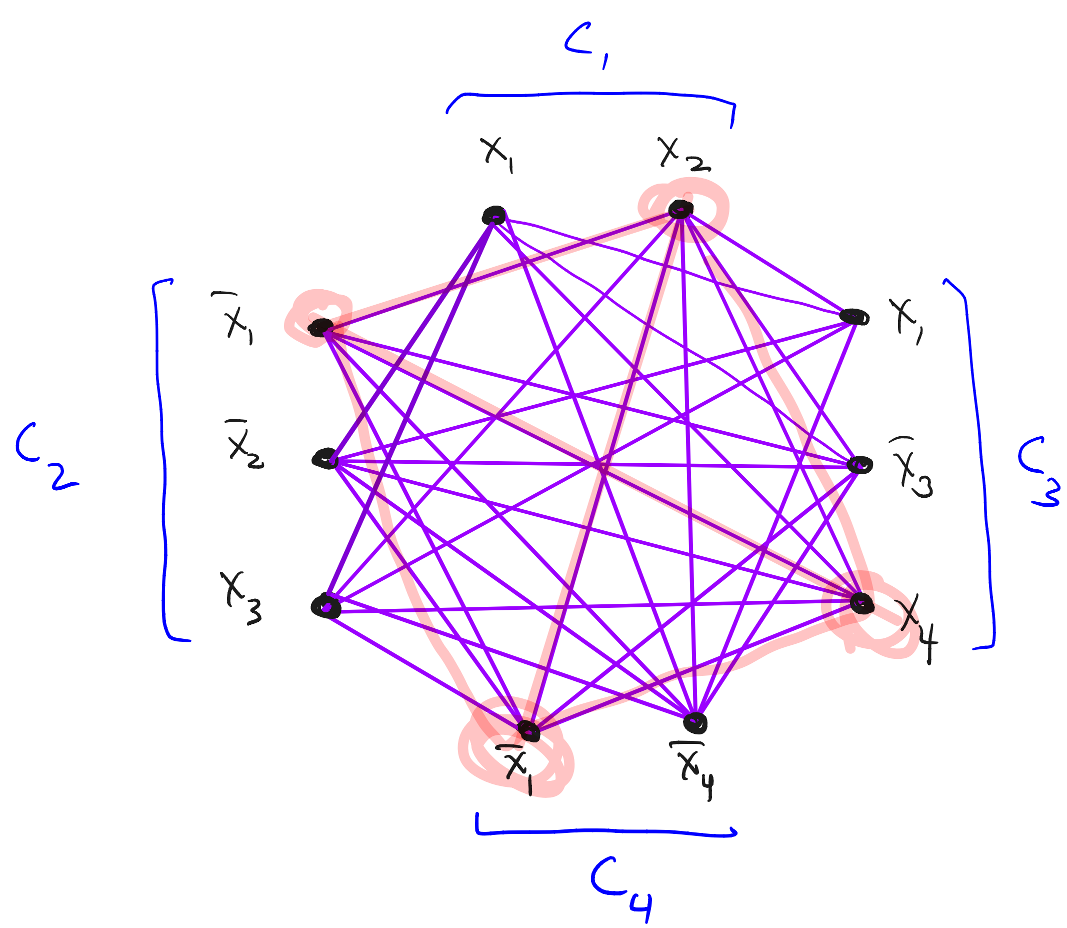
Edges: edge between 2 vertices if they are in different clause groups and if their associated literals are consistent

(all edges present except edges $(x_i, \bar{x}_i)$)
labelled

+ No edges within a clause group

$x = 0$

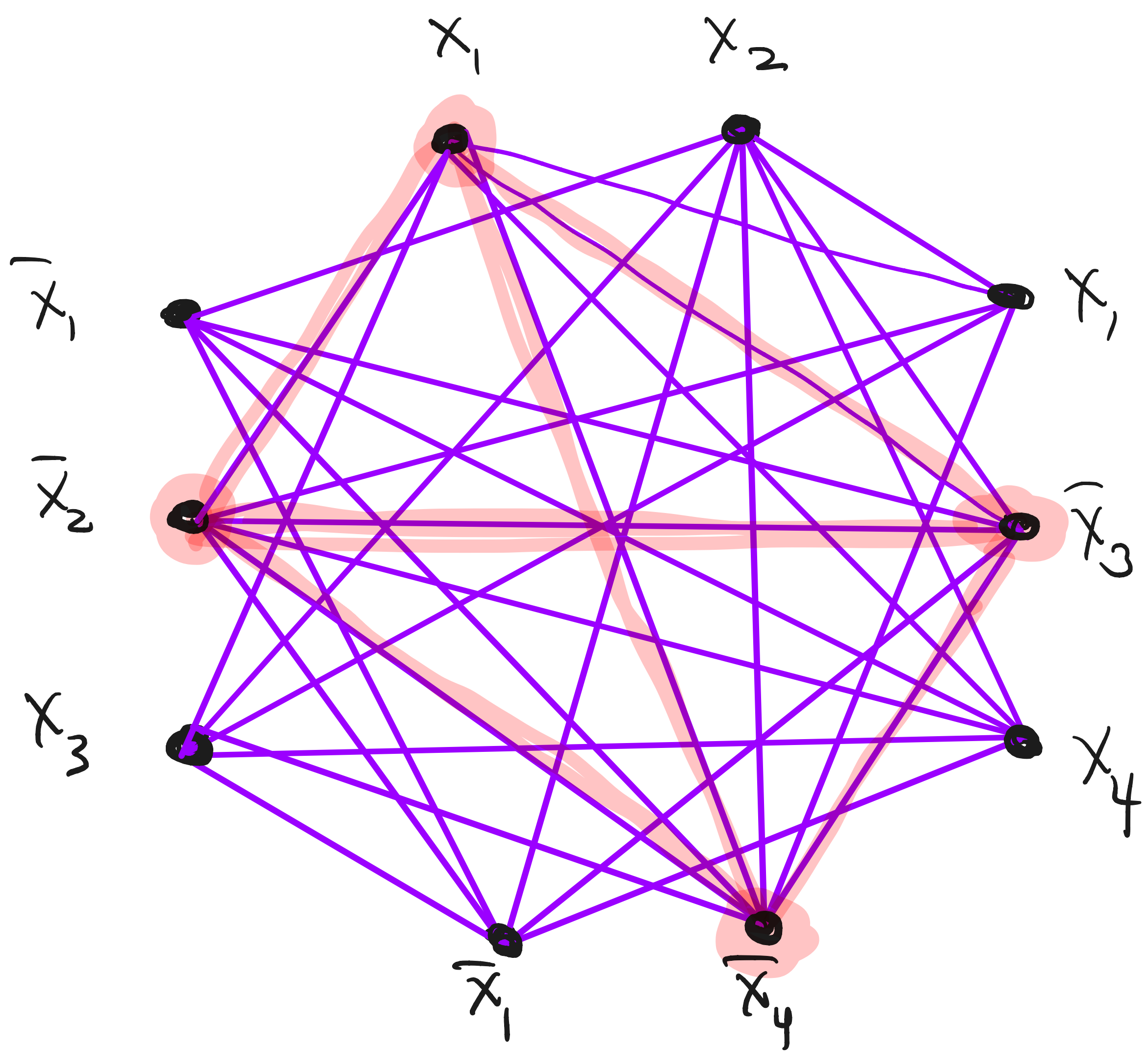
$$\underbrace{\begin{array}{cccc} 0 & 1 & 1 & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \bar{x}_1 & x_2 & x_3 & x_4 \end{array}}$$



$3\text{-SAT} \leq_p \text{CLIQUE}$

Example

Let $\phi = (x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee \bar{x}_3 \vee x_4) \wedge (\bar{x}_1 \vee \overline{x_4})$



edge between 2 vertices if they are in different clause groups and if their associated literals are consistent

α : $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ satisfies ϕ

corresponding 4-clique in red

Correctness

1. Show: if ϕ has a satisfying assignment $\alpha \in \{0,1\}^n$
then G_ϕ has a clique of size m.

Let α be a sat. assignment.
Let ℓ_i be the literal in C_i that is set to true by α

Then the vertices corresponding to $\ell_1, \ldots, \ell_m$
form a clique of size m since ℓ_i, ℓ_j are
consistent

2. Show if G_ϕ has a size m clique then ϕ has a satisfying assignment.

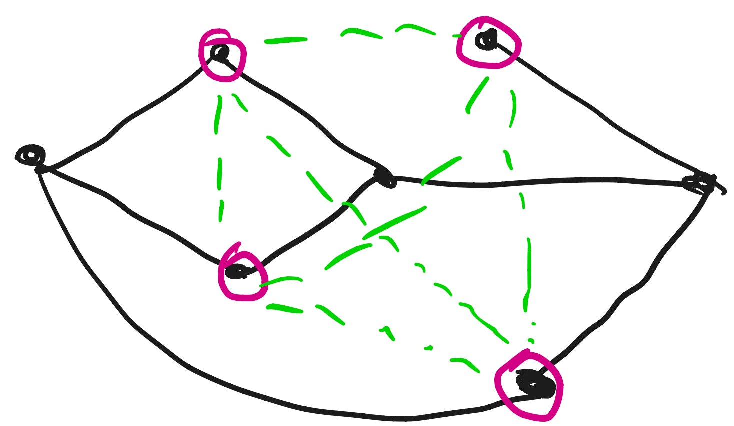
Suppose G_ϕ has a ~~size~~-m clique.
Since no edges are present in a clause group, we must have ~~e~~xactly one vertex from each clause group in the clique.
Let the vertices be labelled $l_1, \ldots, l_m$.
Then $l_1 \ldots l_m$ corresponds to a (partial) assignment of variables (since no (l_i, l_j) are inconsistent)
$\therefore$ this (partial) assignment satisfies all clauses.

(2) Independent set

Input (G, k). G undirected graph

accept iff G contains an indep. set of size K

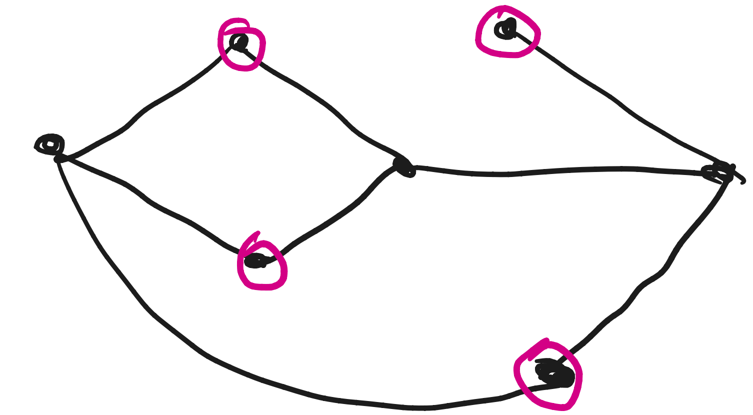
<u>Ex</u>



← has indep set of size 4

does not have size **5** indep. set

② Independent Set



1. Indep Set $\in$ NP

2. Ind set is NP-hard

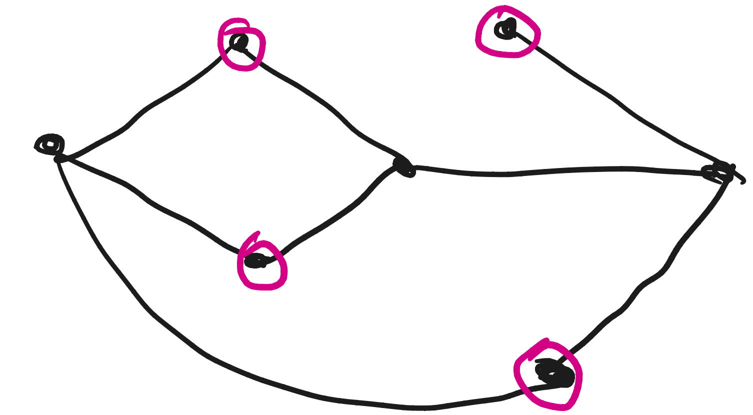
__Pf:__ Show Clique $\leq_p$ Ind-Set (since Clique is NP hard)

$f: (G=(V,E), k) \longrightarrow (G', k)$ $\quad G'=(V, E')$

G': same vertices as G

edge $(i,j) \in E'$ iff $(i,j) \notin E$

② Independent Set



1. Indep Set $\in$ NP

(very similar to Clique $\in$ NP)

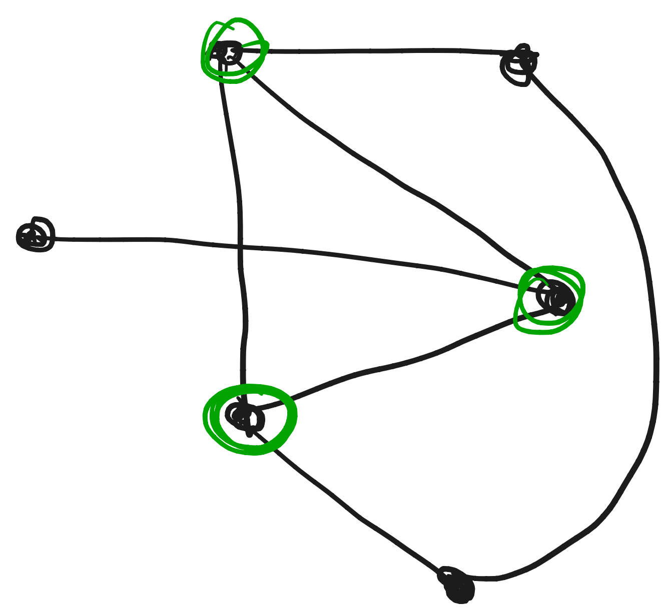
2. Ind set is NP-hard

Pf: Show Clique $\leq_p$ Ind-Set (since Clique is NP hard)

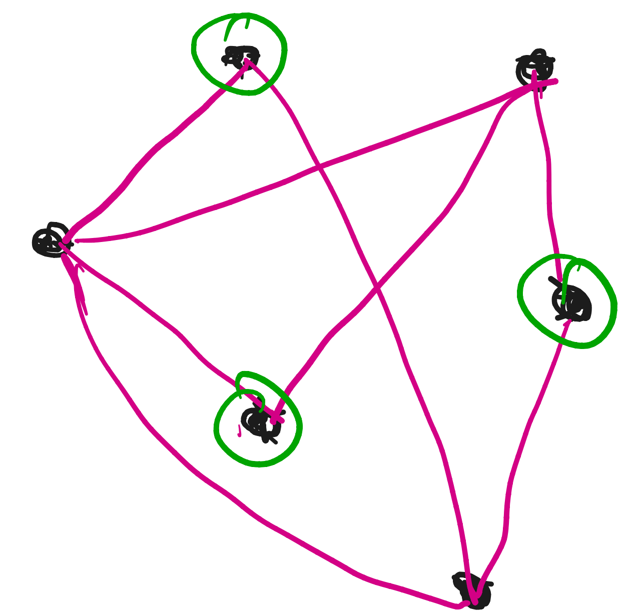
$f : (G = (V, E), k) \longrightarrow (G', n-k) \quad G' = (V, E')$

G' : same vertices as G

edge $(i,j) \in E'$ iff $(i,j) \notin E$



G, k



G', k

<u>Claim</u> G has a clique on $S \subseteq V$ $\iff$ G' has an indep. set on $S \subseteq V$

③ Hitting Set

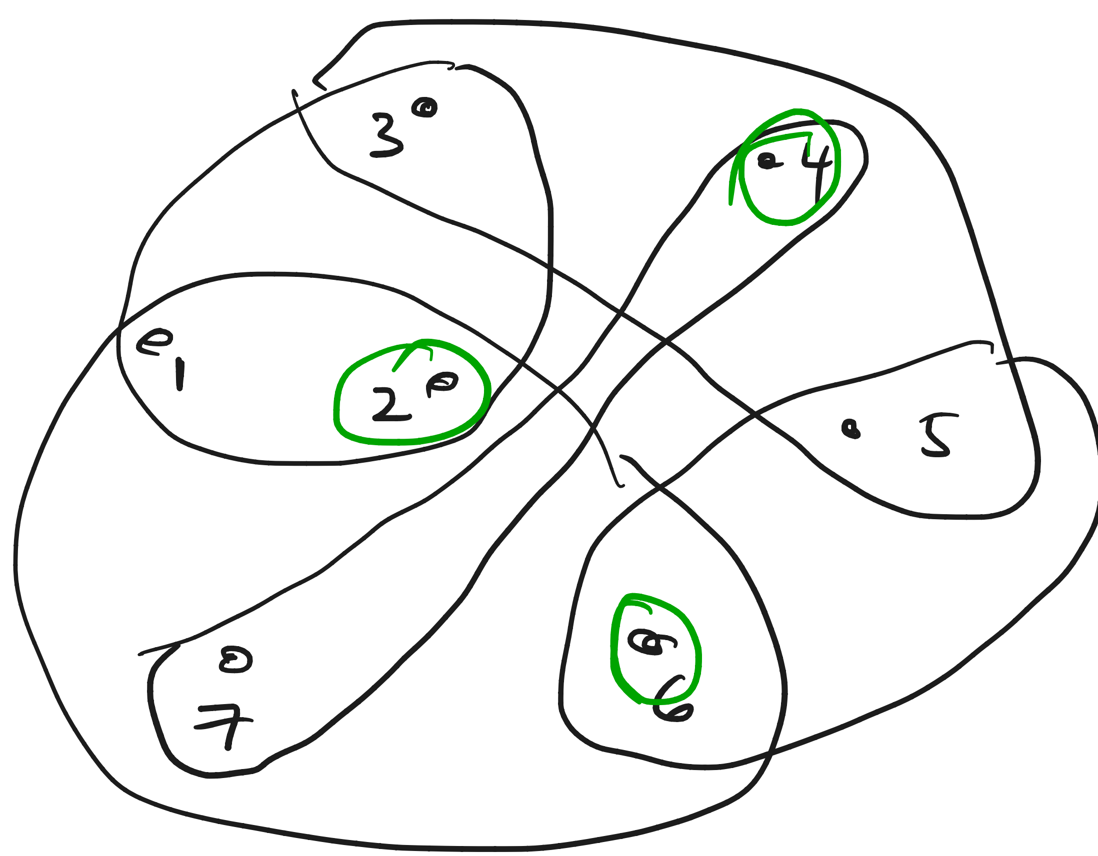
U is universe

Input: a set system $S = \{S_1, \dots, S_m\}$, $S_i \subseteq U$, and K

Accept iff S has a hitting set of size $\le K$

Example

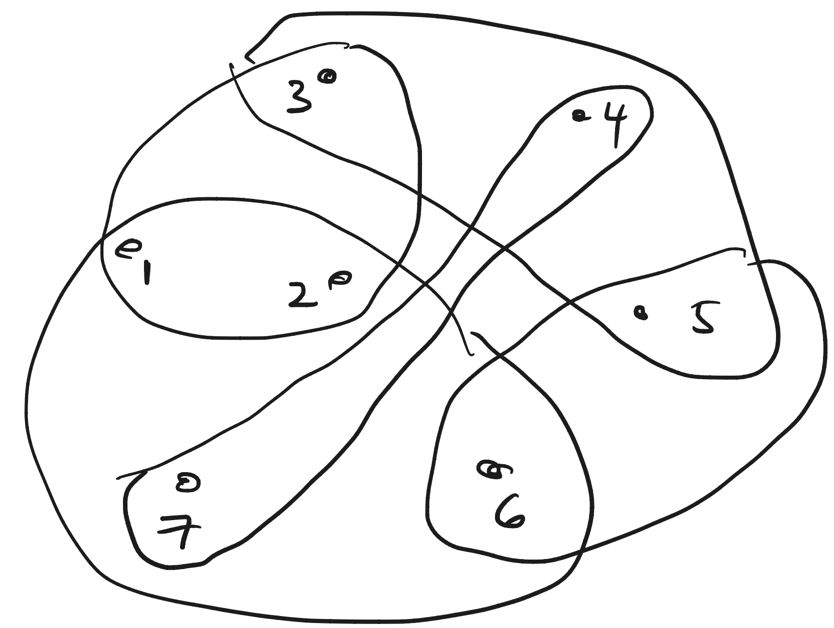
Here $U = \{1, 2, \dots, 7\}$



$$S = \{ S_1 = \{3, 4, 5\},\ S_2 = \{1, 2, 3\},\ S_3 = \{1, 2, 6, 7\},\ S_4 = \{5, 6\},\ S_5 = \{4, 7\} \}$$

$K = 3$

③ Hitting Set



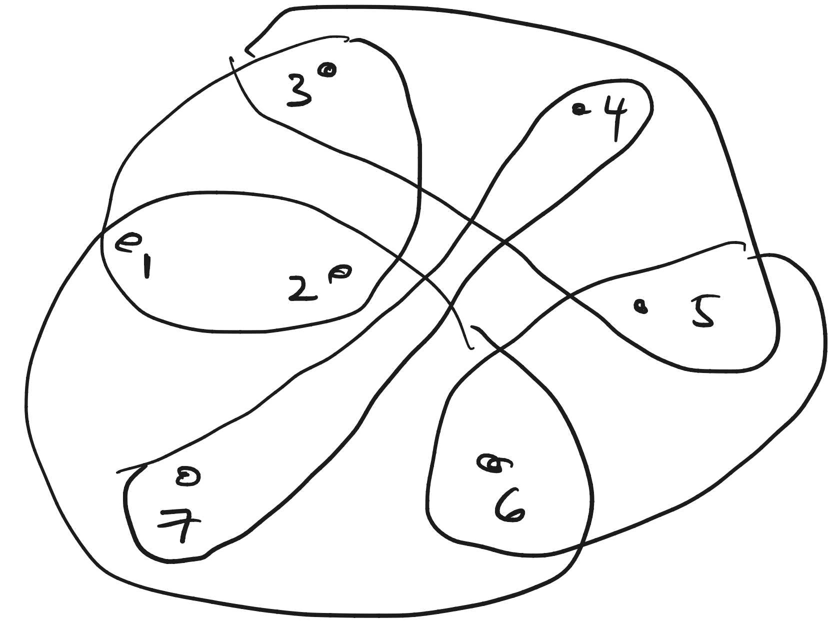
$S_1 = \{3, 4, 5\}$
$S_2 = \{1, 2, 3\}$
$S_3 = \{1, 2, 6, 7\}$
$S_4 = \{5, 6\}$
$S_5 = \{4, 7\}$

Hitting Set in NP:

guess set $H \subseteq [n]$, $|H| = k$

check if $\forall S_i \in S$, $\exists x \in H$ s.t. $x \in S_i$

3 Hitting Set



$$S = \{ S_1 = \{3,4,5\},\ S_2 = \{1,2,3\},\ S_3 = \{1,3,6,7\},\ S_4 = \{5,6\},\ S_5 = \{4,7\} \}$$

Hitting Set NP-hard:

We will show $3SAT \le_p$ Hitting set

given input $C_1 \wedge \ldots \wedge C_m$ to 3SAT on $x_1 \ldots x_n$

create instance of Hitting Set

③ Hitting Set : Input to H.S.
Universe $\mathcal{U}$ (set of vertices)
$S = \{S_1, \dots, S_m\}$, $S_i \subseteq \mathcal{U}$, and K

Ex. $\phi = \underbrace{(x_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_1 \vee x_2)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee x_4)}_{C_3} \wedge \underbrace{(\bar{x}_2 \vee x_3)}_{C_4}$

Hitting set instance : H_ϕ : $|\mathcal{U}| = 2n$ $\quad \mathcal{U} = \{x_1, \dots, x_n, \bar{x}_1, \dots \bar{x}_n\}$

Sets : $\{ \{x_i, \bar{x}_i\} \ \forall\, i = 1, \dots, n \}$

$n = \#$ vars in ϕ
$m = \#$ clauses in ϕ

plus

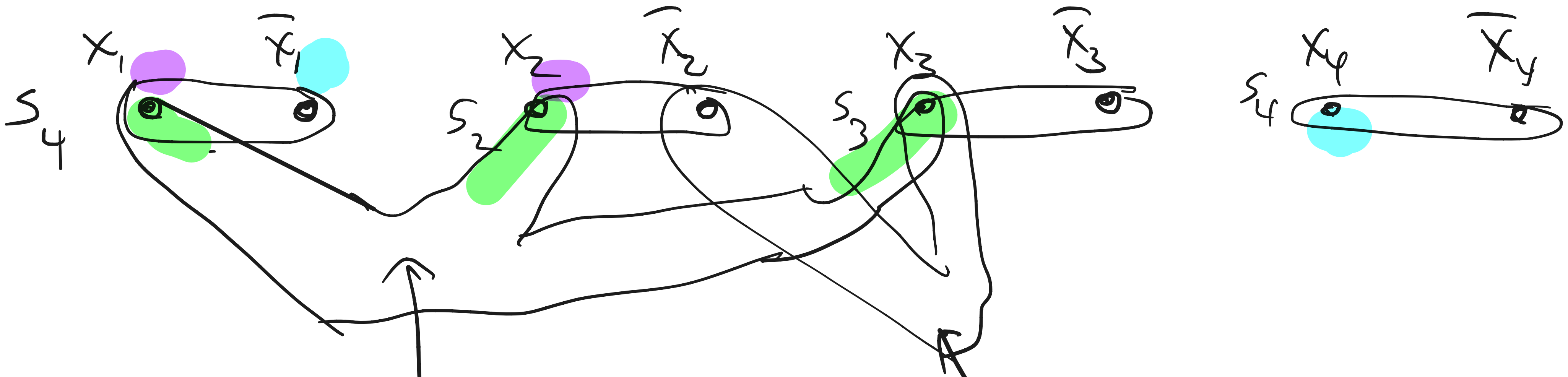
$\{ \mathrm{Set}(C_i) \ \forall\, i = 1, \dots, m \}$

$\mathrm{set}(C_i)$ = set of literals in C_i

③ Hitting Set

$n = 4$

$$\phi = \underbrace{(x_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_1 \vee x_2)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee x_4)}_{C_3} \wedge \underbrace{(\bar{x}_2 \vee x_3)}_{C_4}$$



$\mathrm{Set}(C_1) = \{x_1, x_2, x_3\}$

$\mathrm{Set}(C_4) = \{\bar{x}_2, x_3\}$

S_1

S_2

S_3

S_4

# of underlying elements in set system is $2n$

size of hitting set k:

$k = n = 4$

sets $= n + m$

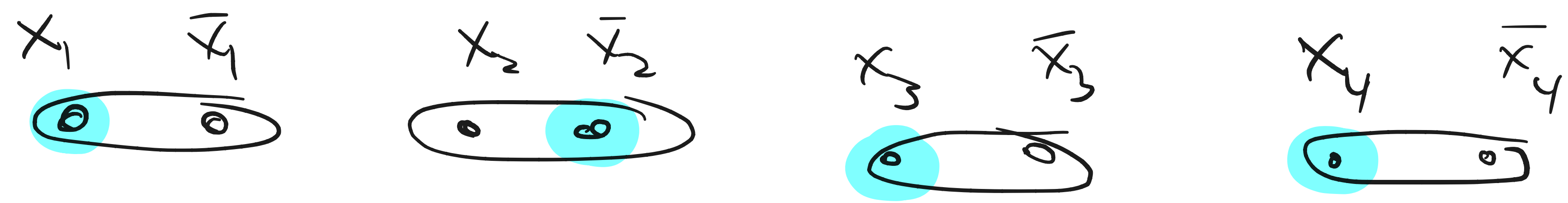
$m =$ # clauses in f

$$\underbrace{\phi}_{\text{input to 3SAT}} \xrightarrow{f} \underbrace{\left(S = \left\{\{x_1, \bar{x}_1\}, \{x_2, \bar{x}_2\}, \{x_3, \bar{x}_3\}, \{x_4, \bar{x}_4\}\right\} \cup \{\mathrm{Set}(C_i) \mid i \in 1 \dots m\},\ k = n\right)}_{\text{input to Hitting-Set}}$$

③ Hitting Set

$n = 4$

$$f = \underbrace{(x_1 \lor x_2 \lor x_3)}_{C_1} \land \underbrace{(x_1 \lor x_2)}_{C_2} \land \underbrace{(\bar{x}_1 \lor x_4)}_{C_3} \land \underbrace{(\bar{x}_2 \lor x_3)}_{C_4}$$



# of underlying elements in set system is $2n$

size of hitting set k: $k = n = 4$

sets $= n + m$

$m =$ #clauses in f

Let $x_1 = 1 \quad x_2 = 0 \quad x_3 = 1 \quad x_4 = 1$

$\text{set}(C_1) = \{x_1, x_2, x_3\}$

$\text{set}(C_2) = \{x_1, x_2\}$

$\text{set}(C_3) = \{\cancel{\bar{x}_1} \lor \cancel{x_4})$

$= \{\bar{x}_2, x_3\}$

Our f : 3SAT $\longrightarrow$ $\underbrace{\text{Hitting set}}$.

$f = C_1 \wedge \dots \wedge C_m$

vars $x_1 \dots x_n$

Universe for H.S. : $U = \{1, \dots, 2n\}$

$\Rightarrow \{x_1\ \bar{x}_1\ x_2\ \bar{x}_2\ \dots\ x_n\ \bar{x}_n\}$

$S = \{ \{x_1\ \bar{x}_1\}, \{x_2, \bar{x}_2\}, \dots, \{x_n, \bar{x}_n\},$ $\longleftarrow n$

$\forall C_i$ set$(C_i) = \{$ set of literals in $C_i\}$ $\}$ $\longleftarrow m$

# of sets in $S = n + m$

pick $K = n$

Other good NP-complete examples

4. Vertex cover

5. Subset ~~S~~ sum (harder)

6. ~~Co~~ graph 3-colorability

7. Variations of all problems

→ exercise

→ Lots of SAT variants

Find a sat. assignment such that every ~~no~~ clause ~~has~~ ~~more~~ contains exactly one satisfied literal.

2SAT $\in P$

Horn SAT $\in P$

Ex.

Exact 3SAT:

input $f = C_1 \wedge \ldots \wedge C_m$ CNF formula

now every clause has size exactly 3

accept f iff f is satisfiable