Lecture 18

- HW4 due Thurs at 11am (only one late HW allowed!)
- See exercise solutions (posted!)

HW 4

For proofs of decidable / recognizable / undecidable :

Explain :

(1) If input $\in L$ then accept

(2) If input $\notin L$ then doesn't accept

We need to see both explanations

HW 4

Last class: we defined Turing reductions (Chap 6.3)

Ex $\quad A_{TM} \leq_T B$

Book first defines a more restricted type of reduction called a mapping reduction (section 5.3)
We will skip this (since Turing reductions more general)

What Sections to read in Book

Can skip:

Chap 5 "Reductions via computation Histories" pp. 220 – 226

Chap 5 Undecidability of Post's corresp. Problem pp 227 – 233

Read:

Chap. 4: (especially 4.2)

Chap 5:
- 5.1, pp 215 – 220
- 5.3 Mapping Reducibility (useful but optional)

Chapter 6:
- 6.3 Turing Reducibility

Today chapter 4.2 + 5.1

goal: Prove Theorem from Lecture 17

Theorem A_{TM} is not decidable

Main idea There are way more languages than TMs

$A_{TM} = \{ \langle M, w \rangle \mid M$ is a TM that accepts input $w \}$

Countable vs Uncountable Sets

Set of TMs are countable

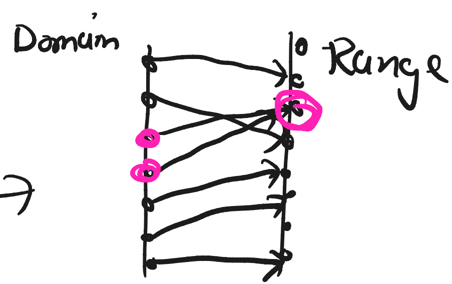
Set of all languages $L \subseteq \{0,1\}^*$ uncountable

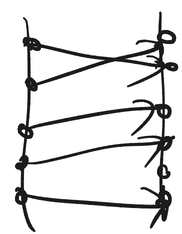
Defn A set S is countable if there exists an injective map f from S to $\mathbb{N}$
(one-to-one)

Injective : $f(x) = f(y) \Rightarrow x = y$

(f never maps 2 different domain elements to same thing)

Not injective →





is injective

Countable vs Uncountable Sets

Set of TMs are countable

Set of all languages $L \subseteq \{0,1\}^*$ uncountable

(1) All finite sets are countable

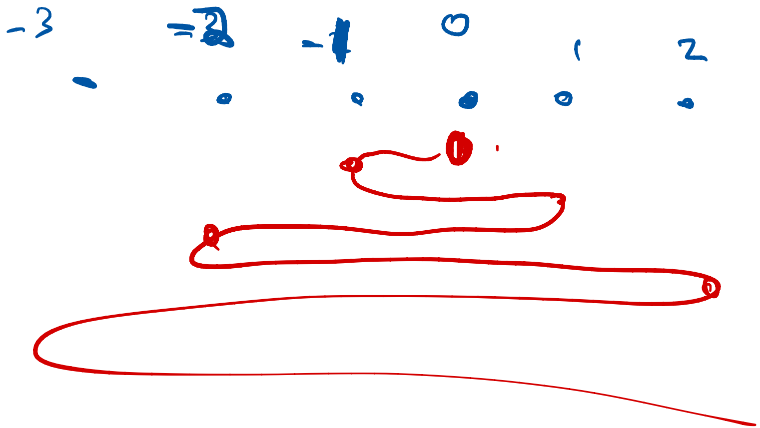
(2) $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, 3, \ldots\}$ is countable

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x-1 & \text{if } x < 0 \end{cases}$$

(3) $\mathbb{N} \times \mathbb{N}$ is countable

$$\begin{array}{cccccccc} (0,0), & (0,1), & (1,0), & (0,2), & (1,1), & (2,0), & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \end{array}$$

	0	1	2	3		
0	0,0	0,1	0,2	0,3	0,4,	...
1	1,0	1,1	1,2	1,3	1,4	...
2	2,0	2,1	3,2	2,3	3,4	...
3	3,0	3,1				


-3
-2
-1
0
1
2

Countable vs Uncountable Sets

Set of TMs are countable

Set of all languages $L \subseteq \{0,1\}^*$ uncountable

(4) The set of all TMs over $\Sigma = \{0,1\}$ is countable

We saw how to represent any M by a <u>unique</u> string $\langle M \rangle \in \{0,1\}^*$

$\therefore$ we can order all M in lexicographic order

$$\begin{array}{cccc} \langle M_1 \rangle, & \langle M_2 \rangle, & \langle M_3 \rangle & \cdots\cdots \\ \downarrow & \downarrow & \downarrow & \\ 1 & 2 & 3 & \cdots \end{array}$$

So the set of all recognizable languages over $\{0,1\}$ is countable.

Countable vs Uncountable Sets

Uncountable sets :

(1) $\mathbb{R}$ (the set of real numbers) ← proved by Cantor, by diagonalization

1834

(2) The set of all languages $L \subseteq \{0,1\}^*$

↑ we'll prove this by diagonalization

Gen real #'s in $[0, 1)$

$0.7543 \ldots$

Many Languages are not r.e. (recognizable)!

Proof Idea Let $\Sigma = \{0,1\}$

- Every TM over $\Sigma = \{0,1\}$ is encoded by a unique string $\langle M \rangle \in \Sigma^*$
- Thus every Turing recognizable language over $\{0,1\}$ can be described/encoded by a string $\langle M \rangle \in \Sigma^*$ (the M that accepts L)
- Thus the set of all Turing-recognizable languages is countable
- But on the other hand the set of all languages over $\{0,1\}$ is uncountable
- Thus most languages $L \subseteq \Sigma^*$ are not recognizable

Theorem There exists a language $L \subseteq \{0,1\}^*$ that is not r.e. (recognizable)

Pf (diagonalization)

Fix an enumeration of all TMs over $\{0,1\}$ using our encoding of TMs

$$M_1, M_2, M_3, \ldots$$

order lexicographically by their encodings (so $\langle M_1 \rangle < \langle M_2 \rangle < \ldots$)

Define $D = \{ \langle M \rangle \mid \langle M \rangle$ encodes TM M, and M on input $w = \langle M \rangle$ does not accept $\langle M \rangle \}$

called the Diagonal Language

$\langle M, \underbrace{w = \langle M \rangle} \rangle$

$\langle M \rangle \in D$ iff M does not accept $w = \langle M \rangle$

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\cdots$
M_1	0	1	0	0	0 $\cdots$
M_2	1	0	1	1	0 $\cdots$
M_3	0	1	1	1	1 $\cdots$
M_4	0	1	0	1	1 $\cdots$
M_5	0	0	1	1	0 $\cdots$
$\vdots$					

entry $(M_i, \langle M_j\rangle)$: $\begin{cases} 1 & \text{if } M_i(\langle M_j\rangle) \text{ halts and accepts} \\ 0 & \text{if } M_i(\langle M_j\rangle) \text{ does not accept} \end{cases}$

Define $D = \{ \langle M \rangle \mid \langle M \rangle$ encodes TM M, and M on input $\langle M \rangle$ does not halt and accept $\}$

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	. . . . . .
M_1	0	1	0	0	0 . . .
M_2	1	0	1	1	0 . .
M_3	0	1	1	1	1 . . .
M_4	0	1	0	1	1 . .
M_5	0	0	1	1	0 . .
⋮					

$\langle M_4 \rangle \notin D$

$\langle M_5 \rangle \in D$

D is the "opposite" of what is on diagonal

That is :
$$D(\langle M_i \rangle) = \begin{cases} 0 & \text{if } M_i(\langle M_i \rangle) \text{ halts and accepts} \\ 1 & \text{otherwise} \end{cases}$$

D: 1 1 0 0 1 . . . .

Real #'s in [0,1] not countable (in book)

x_0 . 0 0 0 . . . —

x_1 . 0 0 0 . . . — — — —

⋮ .

7

4

0

. 0 5 7 9 8 4 . . . — — — —

alleged enumeration of all ~~no~~ reals in (0,1]

r^* : . 1 1 6 3 1 3 . . . .

Theorem There exists a Language $L_D \subseteq \{0,1\}^*$ that is not r.e. (recognizable)

Pf (diagonalization)

Fix an enumeration of all TMs over $\{0,1\}$ using our encoding of TMs

$M_1, M_2, M_3, \ldots$

order lexicographically by their encodings (so $\langle M_1\rangle < \langle M_2\rangle < \ldots$)

Define $D = \{ \langle M\rangle \mid \langle M\rangle$ encodes TM M, and M on input $\langle M\rangle$ does not halt and accept $\}$

called the Diagonal Language

Claim D is not r.e. (recognizable)

Pf By construction, for all TMs M_i over Σ,

$\mathcal{L}(M_i) \neq D$ since $\langle M_i\rangle \in D$ if and only if $\langle M_i\rangle \notin \mathcal{L}(M_i)$

Define $D = \{ \langle M \rangle \mid \langle M \rangle$ encodes TM M, and M on input $\langle M \rangle$ does not halt and accept $\}$

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	. . . . . .
M_1	0	1	0	0	0 . . .
M_2	1	0	1	1	0 . .
M_3	0	1	1	1	1 . . .
M_4	0	1	0	1	1 . .
M_5	0	0	1	1	0 . .
.					
.					

$\langle M_4 \rangle \notin D$

$\langle M_5 \rangle \in D$

Claim D is not r.e. (recognizable)

Pf By construction, for all TMs M_i over Σ,
$\mathcal{L}(M_i) \neq D$ since $\langle M_i \rangle \in D$ if and only if $\langle M_i \rangle \notin \mathcal{L}(M_i)$

Define $\overline{D} = \{ \langle M \rangle \mid \langle M \rangle$ encodes TM M, and M on input $\langle M \rangle$ accepts $\}$

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	...	
M_1	0	1	0	0	0	...
M_2	1	0	1	1	0	...
M_3	0	1	1	1	1	...
M_4	0	1	0	1	1	... $\langle M_4 \rangle \in \overline{D}$
M_5	0	0	1	1	0	... $\langle M_5 \rangle \notin \overline{D}$
⋮						

$\overline{D}$ is what is on the diagonal

That is $\overline{D}(\langle M_i \rangle) = \begin{cases} 1 & \text{if } M_i(\langle M_i \rangle) \text{ accepts} \\ 0 & \text{otherwise} \end{cases}$

__Claim__ $\overline{D}$ is recognizable

__Pf__ : TM for $\overline{D}$ on input $\langle M \rangle$

- check to see if input is legal encoding of a TM
  if not, reject
- otherwise run M on $\langle M \rangle$:
  If simulation halts and accepts $\rightarrow$ halt + accept

Thus we have shown:

$\overline{D}$ is recognizable

D is not recognizable

Question: Is $\overline{D}$ decidable?

By closure property, $\overline{D}$ not decidable

Conclusion:

D not recognizable (so also not decidable)

[$\overline{D}$ is recognizable, but not decidable

Back to A_{TM}: (we want to show A_{TM} not decidable)

Lemma $\overline{D} \leq_T A_{TM}$

Pf: Assume N be a decider for A_{TM}.

Then we build a decider N' for $\overline{D}$:

N' :

On input $\langle M \rangle$:

Run N on $(\langle M \rangle, \langle M \rangle)$

If N accepts $\longrightarrow$ accept $\langle M \rangle$

Otherwise if N rejects $\longrightarrow$ halt + reject $\langle M \rangle$

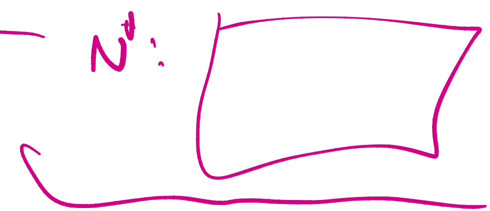
Thm A_{TM} is not decidable

Pf Assume for sake of contradiction A_{TM} is decidable, and let N be a decider for A_{TM}.

Then using N, we can construct a decider, N' for $\overline{D}$

pseudocode for N'

N':



So this shows if A_{TM} is decidable then so is $\overline{D}$,

But $\overline{D}$ not decidable! Contradiction

Theorem A_{TM} not decidable

Proof:

By Lemma if A_{TM} decidable then $\bar{D}$ decidable

But $\bar{D}$ not decidable.

$\therefore$ A_{TM} not decidable !

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1. $D = \{ \langle M \rangle \mid M$ on input $\langle M \rangle$ does not accept$\}$ is not recognizable (by "diagonalization")
2. $\bar{D}$ is recognizable
   $\bar{D}$ not decidable (by closure property)
3. A_{TM} not decidable (by reduction $\bar{D} \leq_T A_{TM}$)

SUMMARY

① $D = \{\langle M\rangle \mid M(\langle M\rangle) \text{ does not accept}\}$ ← Diagonal Language

is not recognizable

Proof by diagonalization

② $\bar{D} = \{\langle M\rangle \mid M(\langle M\rangle) \text{ halts and accepts}\}$

is recognizable but not decidable

③ $A_{TM} = \{\langle M, w\rangle \mid M \text{ accepts } w\}$

is recognizable but not decidable