Lecture 17

HW 4 is out : due Nov 13

Come to OHs for help!

Look at Chapters 4+5 for more examples + TODAY

TIP: Do exercises + lots of practice

proofs are "always the same"

it just takes time / intuition building

Last week: TMs, decidable / recognizable languages

$A_{TM} = \{ \langle M, x \rangle \mid M \text{ accepts } x \}$ ← recognizable

CLOSURE PROPERTIES

① L decidable $\Longrightarrow$ L recognizable

② closure of decidable languages under $\cap, \cup, \neg$:

L_1, L_2 decidable $\Longrightarrow$ $L_1 \cup L_2$, $L_1 \cap L_2$, $\neg L_1$, $\neg L_2$ decidable

③ Closure of recognizable languages under $\cap, \cup$

L_1, L_2 recognizable $\Longrightarrow$ $L_1 \cup L_2$, $L_1 \cap L_2$ are recognizable

* Recognizable languages not closed under complement

④ L, $\bar{L}$ are recognizable $\Longrightarrow$ L is decidable

We'll sketch proof of ④; rest are left as exercises

CLOSURE PROPERTIES

④ L, $\bar{L}$ recognizable $\Rightarrow$ L is decidable

Proof sketch: (Dovetailing)

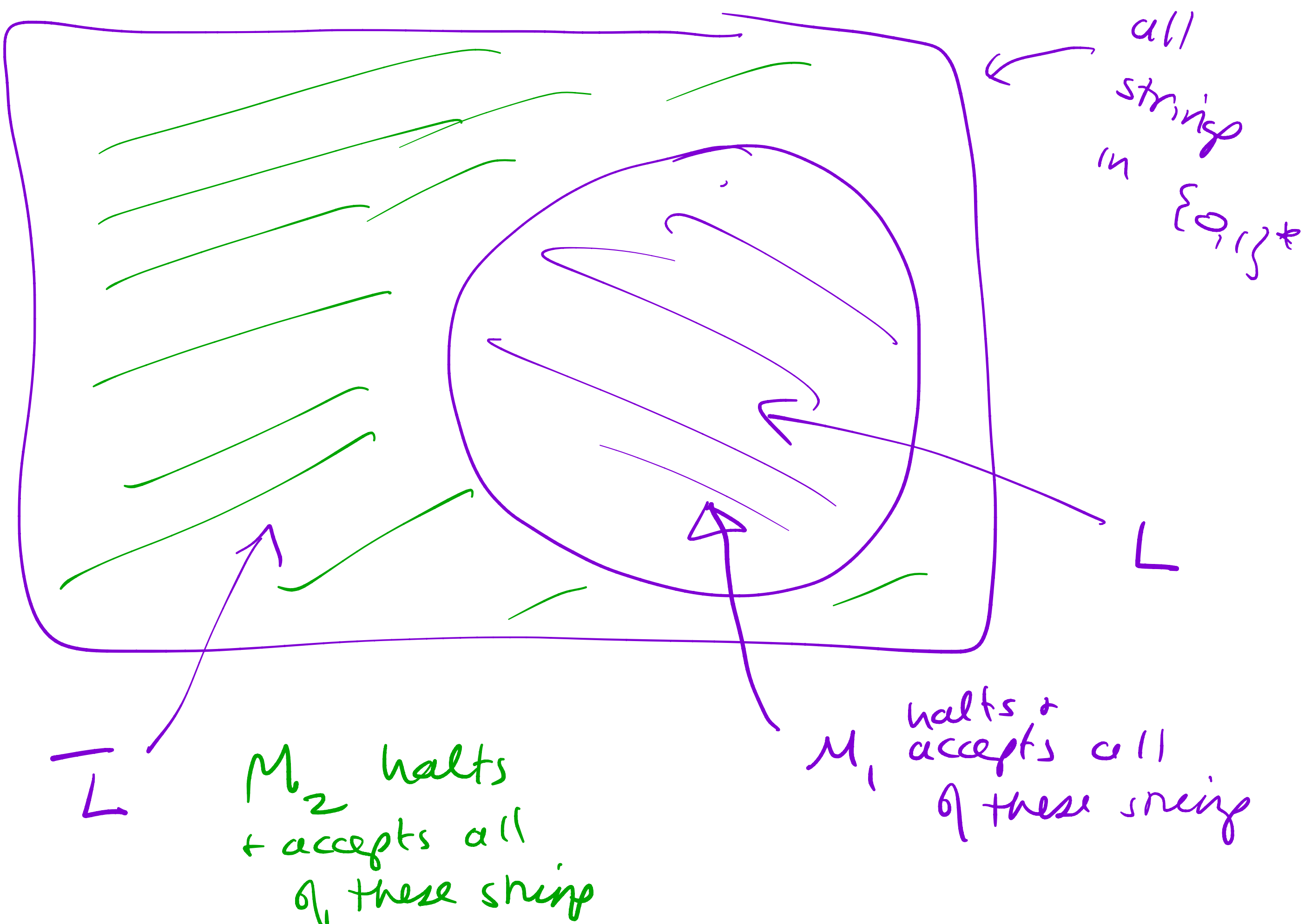
Let M_1 be a TM st $\mathcal{L}(M) = L$ and let M_2 be a TM st $\mathcal{L}(M) = \bar{L}$

New TM M on input x

For $i = 1, 2, 3, \ldots$

- Run M_1 on x for i steps
  - if M_1 accepts x, halt + accept
- Run M_2 on x for i steps
  - if M_2 accepts x, halt and reject

M_1 + M_2 may not always halt.
We want to design M that always halts and such that $\mathcal{L}(M) = L$



all strings in $\{0,1\}^*$
L
$\bar{L}$
M_2 halts + accepts all of these strings
M_1 halts + accepts all of these strings

CLOSURE PROPERTIES

④ L, $\bar{L}$ recognizable $\Rightarrow$ L is decidable

Proof sketch: (Dovetailing)
Let M_1 be a TM st $\mathcal{L}(M) = L$ and let M_2 be a TM st $\mathcal{L}(M) = \bar{L}$

New TM M on x:

(1) For $i = 1, 2, 3, \ldots$
(2) Run M_1 on x for i steps
(3) if M_1 accepts x, halt + accept
(4) Run M_2 on x for i steps
(5) if M_2 accepts x, halt and reject

M_1 + M_2 may not always halt.
We want to design M that always halts and such that $\mathcal{L}(M) = L$

Claim: M always halts and $\mathcal{L}(M) = L$:

$\forall x$ exactly one of $M_1(x)$ and $M_2(x)$ halts and accepts.

$\therefore$ $\forall x$ there is some time step i s.t. either (i) $M_1(x)$ halts and accepts or (ii) $M_2(x)$ halts and accepts

If (i) then $x \in L$ and $M(x)$ halts + accepts (line 3)

If (ii) then $x \notin L$ and $M(x)$ halts + rejects (line 5)

Next: Proving some languages **Not** recognizable / Not decidable

Recall from last week:

$A_{TM} = \{\langle M, x\rangle \mid M \text{ is a TM and } M \text{ accepts } x\}$

We saw that A_{TM} is recognizable

Theorem A_{TM} is Not decidable $\Rightarrow$ Proof Next week

Corollary: $\overline{A_{TM}}$ is Not recognizable

Proof By closure property:

$A_{TM}, \overline{A_{TM}}$ recognizable $\Rightarrow A_{TM}$ decidable

We will use this a lot!

Small Note:

Technically $\overline{A_{TM}} \neq \overbrace{\{ \langle M, x \rangle \mid M \text{ is a TM and } M \text{ doesnt accept } x \}}^{\text{called } \widetilde{A_{TM}}}$

$\overline{A_{TM}}$ also contains all strings w such that w isn't a legal encoding of a pair $\langle M, w \rangle$

We'll assume $\overline{A_{TM}} = \widetilde{A_{TM}}$ since

$\overline{A_{TM}}$ decidable iff $\widetilde{A_{TM}}$ decidable

Example:

$A_{TM} = \{ \langle M, x \rangle \mid$ input is a legal encoding of a TM M and input x and M accepts $x \}$

$\overline{A_{TM}} = \{ w \mid$ either ① w is not a legal encoding of a pair M, x

or ② M doesn't accept $x \}$

we want to ~~B~~ view

$\overline{A_{TM}} = \{ \langle M, x \rangle \mid M$ doesn't accept $x \}$

Showing Languages not decidable by Turing Reductions

<u>Defn</u> Language A is Turing-reducible to language B written $A \leq_T B$ if a decider for B implies a decider for A.

* If $A \leq_T B$ and B is decidable $\Longrightarrow$ A decidable
* If $A \leq_T B$ and A is <u>un</u>decidable $\Longrightarrow$ B <u>un</u>decidable

Example $HALT = \{\langle M, x\rangle \mid M$ is a TM and M halts on input $x\}$

Lemma 1 HALT is recognizable (exercise)

Lemma 2 HALT is not decidable → Pf by contradiction.

Assume HALT is decidable, + let N be a decider for it. Then we will get a contradiction

Proof: Let's show $A_{TM} \leq HALT$:

Assume there is a decider, N for HALT

Build a decider D for A_{TM} using N

D: $\boxed{\begin{array}{l} \text{On input } \langle M, w\rangle: \\ \quad \text{Run } N \text{ on } \langle M, w\rangle \\ \qquad \text{If } N \text{ rejects} \rightarrow \text{halt + reject} \\ \qquad \text{If } N \text{ accepts:} \\ \qquad\quad \text{Run } M \text{ on } w \text{ (Now we know } M \text{ halts on } w) \\ \qquad\quad \text{If } M \text{ accepts} \Rightarrow \text{halt + accept} \\ \qquad\quad \text{If } M \text{ rejects} \Rightarrow \text{halt + reject} \end{array}}$

Turing Reduction from $A_{TM} \leq_T HALT$

Proof: Let's show $A_{TM} \leq HALT$:

Assume there is a decider, N for $HALT$

Build a decider D for A_{TM} using N

D:

On input $\langle M, w\rangle$:

1. Run N on $\langle M, w\rangle$
2. If N rejects $\rightarrow$ halt & reject
3. If N accepts:
4. Run M on w (Now we know M halts on w)
5. If M accepts $\Rightarrow$ halt & accept
6. If M rejects $\Rightarrow$ halt & reject

Correctness:

1. If M accepts $w \Rightarrow N$ accepts $\langle M, w\rangle$, so when we run M on w it accepts, so D accepts (line 5)
2. If M halts & rejects $w \Rightarrow N$ accepts $\langle M, w\rangle$, so when we run M on w it accepts so D halts & rejects (line 6)
3. If M never halts $\Rightarrow N$ rejects, so D halts & rejects (line 2)

Example $E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } \mathcal{L}(M) = \phi \}$

$$\overline{E}_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } \mathcal{L}(M) \neq \phi \}$$

① $\overline{E}_{TM}$ is recognizable (proof uses dove-tailing)

$E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } \mathcal{L}(M) = \phi \}$

(2) E_{TM} is not decidable. Show $A_{TM} \leq_T E_{TM}$

(then since A_{TM} not decidable $\Rightarrow$ E_{TM} not decidable)

Let N be a decider for E_{TM}. We build a decider D for A_{TM}:

D :

on input $\langle M, x \rangle$:
Let M' be a TM that on input w, M' ignores its input and simulates M on x. If M halts on x then M' halts and accepts
Run N on $\langle M' \rangle$
If N rejects $\langle M' \rangle$ $\rightarrow$ halt and accept
otherwise $\rightarrow$ halt and reject

* M' depends on M, x

$E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } \mathcal{L}(M) = \emptyset \}$

(2) E_{TM} not decidable. We'll show $A_{TM} \leq_T E_{TM}$

Let N be a decider for E_{TM}

D: $\boxed{\begin{array}{l} \text{on input } \langle M, x \rangle: \\ \quad \text{Let } M' \text{ be a TM that on input } w,\ M' \\ \quad\quad \text{ignores its input and simulates } M \text{ on } x. \\ \quad\quad \text{If } M \text{ accepts } x \text{ then } M' \text{ halts and accepts} \\ \quad \text{Run } N \text{ on } \langle M' \rangle \\ \quad \text{If } N \text{ rejects } \langle M' \rangle \to \text{halt and accept} \\ \quad \text{otherwise } \to \text{halt and reject} \end{array}}$

M' accepts all strings if M accepts x

M' accepts NO strings if M does not accept x

$\mathcal{L}(M')$ nonempty iff M accepts x

If M accepts x : ~~then~~ $\Leftrightarrow$ $\mathcal{L}(M')$ = all strings so $\langle M' \rangle \notin E_{TM}$ so N rejects $\langle M' \rangle$

If M doesn't accept x : ~~then~~ $\Leftrightarrow$ $\mathcal{L}(M') = \emptyset$ so $\langle M' \rangle \in E_{TM}$ so N accepts $\langle M' \rangle$

Example $\overline{E}_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \neq \emptyset \}$

① $\overline{E}_{TM}$ is recognizable (proof uses dove-tailing)

Recognizer for $\overline{E}_{TM}$: On input $\langle M \rangle$

For $i = 1, 2, 3, \ldots$

Run M on all strings w with $|w| \leq i$ for i steps

Accept if M accepts any of these strings

otherwise

Alternative:

Let $w_0 \quad w_1 \quad w_2 \quad w_3 \quad w_4 \quad \ldots$ be an ~~on~~ enumeration of all inputs in $\{0,1\}^*$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

$\varepsilon \quad 0 \quad 1 \quad 00 \quad 01$

For $i = 0, 1, \ldots$

Run M on inputs $w_0 \ldots w_i$ for i steps

If any halt + accept $\longrightarrow$ HALT + accept

Otherwise

So far we showed

① E_{TM} not decidable

② $\overline{E_{TM}}$ is recognizable

③ $\overline{E_{TM}}$ decidable ? $\longrightarrow$ No since decidable languages closed under negation

④ E_{TM} recognizable ? $\longrightarrow$ No. since

$L, \overline{L}$ are both recog $\longrightarrow$ L is decidable

$\overline{E_{TM}}$: recognizable not decidable

E_{TM} : not decidable not recognizable.