# Presentations

Schedule of presentation dates is posted
(see course webpage)

Signup to discuss your presentation with
me and your group by Mar 1
(see google doc)

## Last Class

Randomized CC LB of $\Omega(n)$ for Inner Product

## Today

- Slicker randomized CC LB for IP (BNS)
- NOF communication:
  - Defn, cylinder intersections
  - Important Functions
  - Applications of NOF CC in additive combinatorics

## Slightly Slicker proof of lower Bound for randomized cc via discrepancy (that generalizes to NOF)

Called "BNS" method after seminal paper by Babai, Nisan, Szegedy introducing NOF cc and this method

**Theorem [BNS bound]**

Let $f: X \times Y \to \{-1,1\}$, $\mu$ = unif distrib over $X \times Y$

Then $\forall R = S \times T$ $\quad \mathrm{Disc}_\mu(R, f)^2 \leq \mathbb{E}_{Y,Y'} \left| \mathbb{E}_{X} M_f(x,y) \cdot M_f(x',y) \right|$

Theorem [BNS bound] $\quad R = S \times T$

$$\mathrm{Disc}_\mu(R,f)^2 \le \mathop{\mathbb{E}}_{Y,Y'} \left| \mathop{\mathbb{E}}_{X} M_f(x,y) M_f(x',y) \right| \quad , \quad \mu = \text{uniform distrib over } X \times Y$$

Proof ($\mu$ = unif. distrib)

$$\mathrm{Disc}_\mu(f, S\times T) = \left| \mathop{\mathbb{E}}_{x,y} 1_S(x)\, 1_T(y)\, M_f(x,y) \right|$$

so
$$\mathrm{Disc}_\mu(f, S\times T)^2 = \left( \mathop{\mathbb{E}}_{x,y} 1_S(x)\, 1_T(y)\, M_f(x,y) \right)^2$$

$$= \left( \mathop{\mathbb{E}}_{x} 1_S(x) \mathop{\mathbb{E}}_{y} 1_T(y)\, M_f(x,y) \right)^2$$

Cauchy Swartz
$$(\mathbb{E}[Z])^2 \le \mathbb{E}[Z^2]$$

$$\le \mathop{\mathbb{E}}_{x} \left( 1_S(x) \mathop{\mathbb{E}}_{y} 1_B(y) M_f(x,y) \right)^2$$

replace $S$ by $S = \{0,1\}^n$

$$= \mathop{\mathbb{E}}_{x} \left( \mathop{\mathbb{E}}_{y} 1_B(y)\, M_f(x,y) \right)^2$$

$$= \mathop{\mathbb{E}}_{x} \left( \mathop{\mathbb{E}}_{y,y'} 1_B(y)\, 1_B(y')\, M_f(x,y)\, M_f(x,y') \right)$$

$$= \mathop{\mathbb{E}}_{y,y'} 1_B(y)\, 1_B(y') \left( \mathop{\mathbb{E}}_{x} M_f(x,y)\, M_f(x,y') \right)$$

$$\le \mathop{\mathbb{E}}_{y,y'} \left| \mathop{\mathbb{E}}_{x} M_f(x,y)\, M_f(x,y') \right| \quad \blacksquare$$

Theorem [BNS bound] $R = S \times T$

$$\mathrm{Disc}_\mu(R,f)^2 \le \mathop{\mathbb{E}}_{y,y'} \left| \mathop{\mathbb{E}}_{x} M_f(x,y) M_f(x',y) \right| \quad , \quad \mu = \text{uniform distrib over } X \times Y$$

Theorem 2 $\mathrm{DISC}_\mu(IP_n, S\times T) \le 2^{-n/2}$ [and thus by Theorem 1, $D_\mu^{1/3}(IP_n) = \Omega(n)$]

Pf :

$$\mathrm{Disc}_\mu(IP_n, S\times T)^2 \le \mathop{\mathbb{E}}_{y,y'} \left| \mathop{\mathbb{E}}_{x} M_{IP}(x,y) M_{IP}(x,y') \right| \qquad \text{[By BNS Bound]}$$

$$= \Pr[y = y'] = 2^{-n}$$

[orthogonality of rows of $H_n$ :

$$\mathop{\mathbb{E}}_{x} M_{IP}(x,y)\, M_{IP}(x,y') = \begin{cases} 0 & \text{if } y \ne y' \\ 1 & \text{if } y = y' \end{cases}$$
]

$$\therefore \mathrm{Disc}_\mu(IP_n) = 2^{-n/2}$$

$\blacksquare$

The BNS Bound $\mathrm{Disc}_\mu(R, f)^2 \le \mathop{\mathbb{E}}_{y,y'} \left| \mathop{\mathbb{E}}_{x} M_f(x,y)\, M_f(x,y') \right|$ holds

for any product distribution $\mu$

($\mu = \mu_1 \times \mu_2$ where $\mu_1$ is distrib on $X$, $\mu_2$ on $Y$)

with same prob.

# Multiparty CC

There are 2 different models when $k$ (# players) $\geq 3$

(they coincide when $k=2$)

(1) NIH (Number IN Hand)

Player 1 gets $x_1$
2 $x_2$
3 $x_3$

$\Bigg\}$ Typically $|x_1| = |x_2| = |x_3|$

(2) NOF (Number on Forhead)

Player 1 sees $x_2, x_3$
2 " $x_1, x_3$
3 " $x_1, x_2$

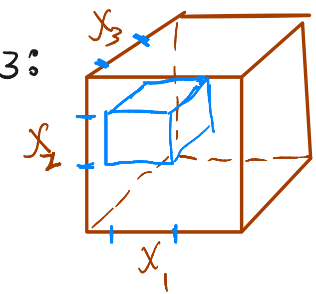
# NIH Protocols $k \geq 2$

A $k$-party NIH protocol $\Pi$ for $f(x_1, \ldots, x_k)$, each $|x_i| = n$

- Player $i \in [k]$ sees $x_i$
- Just like 2-party case, a $k$-party $\Pi$ specifies
  - which player's turn it is to speak (all msgs are broadcast to all players)
  - Fxn at nd $r$ (assuming Player $i$ speaks) depends on transcript + $x_i$

- Deterministic $k$-party cc: $D_k^{NIH}(f)$
  Deterministic $\Pi$ for $f(x_1 \ldots x_k)$ partitions $X_1 \times \cdots \times X_k$ into monochromatic $k$-dimensional rectangles (tensors)

$k=3$:




- We can also define randomized NIH and Distributional complexity wrt. distrib $\mu$ over $X^1 \times \ldots \times X^k$ (analogous to $k=2$)

We'll mostly focus on NOF CC due to its many applications, including:

* (1) Additive Combinatorics

(2) ACC Lower bounds

(3) Proof Complexity Lower Bounds

* (4) Matrix Multiplication

* : Presentation Topic

* Today

# NOF Protocols

A k-party NOF protocol $\Pi$ for $f(x_1, \ldots, x_k)$, each $|x_i| = n$

- Player $i \in [k]$ sees all $x_j$ except for $x_i$
- Just like 2-party case, a k-party $\Pi$ specifies
  - which player's turn it is to speak (all msgs are broadcast to all players)
  - Fxn at rnd $r$ (assuming Player $i$ speaks) depends on: transcript so far, and inputs $X^{-i} \triangleq \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\}$

Deterministic k-party cc: $D_k(f)$

Distributional complexity wrt. distrib $\mu$ over $X_1 \times \ldots \times X_k$ : $D_k^{\varepsilon, \mu}(f)$

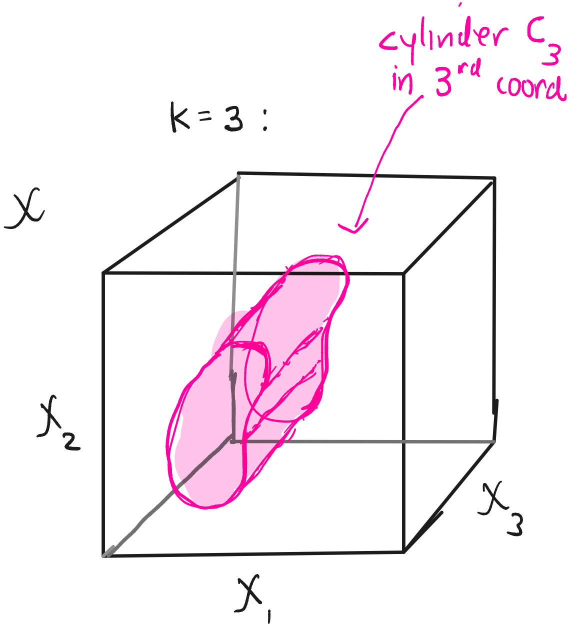
# NOF Protocols

**Defn.** Let $X = X_1 \times X_2 \times \cdots \times X_k$ (input space)

A cylinder $C_i$ in $i^{th}$ coordinate is a subset of $X$ that doesn't depend on $i^{th}$ coordinate

ie. $(x_1, \ldots, x_i, \ldots x_k) \in C_i \Rightarrow \forall x_i' \in X_i : (x_1, \ldots, x_i', \ldots x_k) \in C_i$

K = 3 :




**(Easy) Proposition** If $C_i, C_i'$ are cylinders in $i^{th}$ coord, then so is $C_i \cap C_i'$

**Defn** A cylinder intersection is a subset

$C \subseteq X$ s.t. $C = C_1 \cap C_2 \cap \cdots \cap C_k$ , where $C_i$ = cylinder intersect in $i^{th}$ coord.

**Claim** Let $\Pi$ be a cost $c$ NOF protocol.
then $\Pi$ induces a partition of $X$ into $2^c$ cylinder intersections
(pf by induction on $c$, using Proposition above)

Analog of 2-party partition into rectangles

# Important NOF Functions

(1) gIP (generalized Inner Product)

Input $\vec{x}_1, \ldots, \vec{x}_k$ (Player $j \in [k]$ sees all $\vec{x}_{j'}$, $j' \neq j$)

$gIP(\vec{x}_1, \ldots, \vec{x}_k) = 1$ iff $\sum_{i=1}^{n} x_{1,i} \wedge \cdots \wedge x_{k,i} = 1 \bmod 2$

(ie output is 1 iff size of intersection $x_1 \cap \cdots \cap x_k = 1 \bmod 2$)

(2) k-player Disjointness

$DISJ(x_1 \ldots x_k) = 1$ iff $\sum_{i=1}^{n} x_{1i} \wedge \cdots \wedge x_{ki} \geq 1$ ← easy for nondet protocol

- For k = constant both gIP, DISJ require randomized comm. $\Omega(n/4^k)$

# Important NOF Functions

(3) Exactly-N function ($k=3$)

Each player $i \in \{1,2,3\}$ gets a number $x_i \in [N]$ $\left( \begin{array}{l} |x_i| = n \\ N = 2^n \end{array} \right)$

Output 1 iff $x_1 + x_2 + x_3 = N$

↑

Has $O(1)$-cost randomized NOF protocol!

Longstanding open problem: Best deterministic NOF protocol?

Deterministic NOF cc of Exactly-N ($k=3$) is essentially <u>equivalent</u> to corners problem in additive combinatorics

# Important NOF Functions

(3) Exactly-N function ($k=3$)

Each player $i \in \{1,2,3\}$ gets a number $x_i \in [N]$ $\left( \begin{array}{l} |x_i| = n \\ N = 2^n \end{array} \right)$

Output 1 iff $x_1 + x_2 + x_3 = N$

## Randomized protocol

Player 1 sees $x_2, x_3$ and computes $x_2 + x_3$

Player 2 sees $x_1$ and computes $N - x_1$

Players 1, 2 run 2-party randomized EQ protocol to check if $N - x_1 = x_2 + x_3$

cost $= O(1)$ !

## NOF model : some surprising UPPER BOUNDS

(1.) $EQ(x_1, \dots, x_k) = 1$ iff $x_1 = \dots = x_k$

Note $k=2$ : det cc is $\Omega(n)$

But easy for $k > 2$!

Player 1 : check if $x_2 = \dots = x_k$
If not halt + output 0
ow send 1

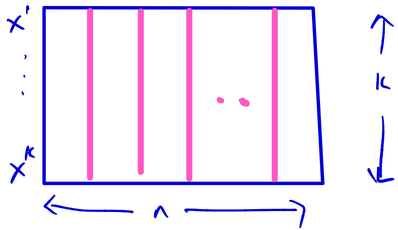
Player 2: check if $x_1 = x_3$.
output 1 if equal + halt
ow output 0 + halt

(2). $D^k(GIP^k_n) = O\left(\frac{kn}{2^k}\right)$ [Grolmusz '94]

kPlayer
Determin. NOF

$$\boxed{D^k(\mathrm{gip}^k_n) = O\left(\frac{kn}{2^k}\right)}$$

Protocol:

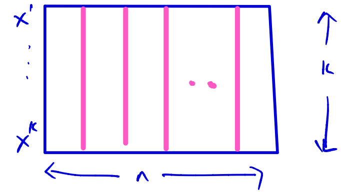



$\leftarrow$ Input to $\mathrm{gip}^k_n$

- Divide columns into blocks of size $2^{k-1}-1$
- Compute gip for every block & then sum results mod 2 to get answer

  cost $\approx \frac{n}{2^k} \cdot$ cost-per-block

$$\boxed{D^k(gip^k_n) = O\left(\frac{kn}{2^k}\right)}$$

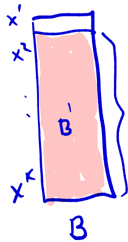



← Input to $gip^k_n$

Protocol:

- Protocol for a block $B$:

1. P1 finds a vector $\alpha \in \{0,1\}^k$ that is not in columns($B$), & sends to other players




Player 1 sees these $k-1$ rows

By PHP $\exists$ vector $\alpha'$ not in columns($B'$)

extend $\alpha' \to \underbrace{1\alpha}_{\alpha}$

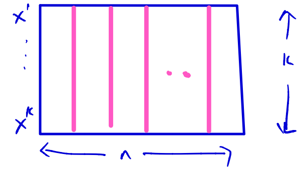
2. If $\alpha$ = all-1 vector $\Rightarrow$ output $\neg$

Else $\alpha$ contains at least one 0. Players want to compute # of all-1 column vectors in $B$

assume $\alpha$ has $\ell$ 0's and $k-\ell$ 1's (for some $\ell$)

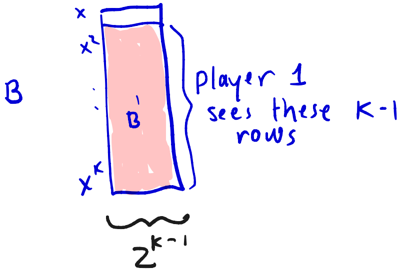
Let $Y_i$ = # column vectors in $B$ with $i$ 0's, $n-i$ 1's

$$\boxed{D^k(gip^k_n) = O\left(\frac{kn}{2^k}\right)}$$




← Input to $gip^k_n$

**Protocol:**




$gip(B) = \#$ all-1 columns in B (mod 2)

Suppose $\alpha$ has $\ell$ 0's, $k-\ell$ 1's: $\underbrace{00000\cdots0}_{\ell}\ \underbrace{11\cdots1}_{k-\ell}$ ← permute rows so all 0's then all 1's

$\forall i \le \ell$: $Y_i \stackrel{d}{=} \#$ column vectors of form $\underbrace{0\cdots0}_{i}\ \underbrace{111\cdots1}_{k-i}$

$Z_i \stackrel{d}{=} \#$ column vectors of form $\underbrace{00\cdots0}_{i-1} * \underbrace{111\cdots1}_{k-i}$

**Claim** $Z_i = Y_{i-1} + Y_i$ ← player $i$ can compute $Z_i$!

$\Pi$:
- Players $i \in [1,\dots,\ell]$ compute + send $Z_i$
- They all know $Y_\ell = 0$

From $Z_1,\dots,Z_\ell, Y_\ell$ they can compute $Y_0$

**Cost:** $\frac{n}{2^k} \cdot k$

# (Explicit) NOF Lower Bounds - What is known

1. GIP : LB $\Omega\left(\frac{n}{4^k}\right)$ randomized k-player NOF

   breaks down when $k > \log n$

   * No Explicit LBs in NOF model for $k > \log n$

We'll prove slightly weaker LB due to BNS
Babai, Nisan, Szegedy

2. k-player Disjointness : LB $\Omega\left(\frac{n}{4^k}\right)$ randomized k-player

3. What about functions that are "easy" for randomized NOF ?
   Like Exactly N ?

Many open questions here in NOF CC are actually <u>equivalent</u> to fundamental problems in additive combinatorics

# (Explicit) NOF Lower Bounds - What is known

3. What about functions that are "easy" for randomized NOF ?
   Like ExactlyN ?

## Recent breakthrough LBs for deterministic NOF $k=3$

(i) "Strong Bounds for 3-progressions" [Kelley, Meka FOCS'23]
gives $\Omega(n^{\varepsilon})$ LB on NIH PromiseEQ $k=3$

(ii) "Explicit separations between randomized & deterministic NOF comm."
[Kelley, Lovett, Meka STOC'24]

(iii) "Quasipoly LBs for the corners theorem"
[Jaber, Liu, Lovett, Ostuni, Sawhney '25]

gives $\Omega(n^{\varepsilon})$ deterministic NOF LB for ExactlyN $k=3$

# Some Equivalences between Additive Comb's Problems + NOF CC Problems

[we will do case of $k=3$ but equivalences hold $\forall k \geq 3$]

① <u>3-AP Problem</u>: What is max size $r_3(N)$ of a subset $S \subseteq [N]$ s.t. $S$ does not contain a 3AP?

3AP: $(x, x+y, x+2y) \in [N]^3$ )


① <u>3-AP Coloring Problem</u>: What is min $c_3(N)$ s.t. $[N]$ can be partitioned into $c_3(N)$ subsets s.t. no subset contains a 3AP?

3AP: $(x, x+y, x+2y) \in [N]^3$ )

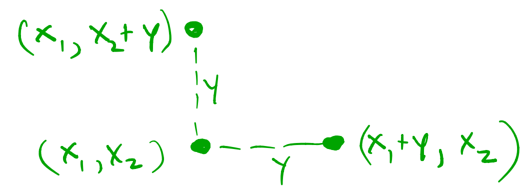

* It is known that ① $\cong$ ① $\left( c_3(N) \approx \dfrac{N}{r_3(N)} \right)$

# Some Equivalences between Additive Comb's Problems + NOF CC Problems

[we will do case of $k=3$ but equivalences hold $\forall k \geq 3$]

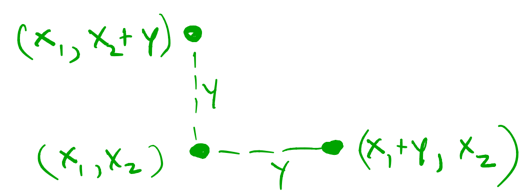
(2) <u>Corners Problem</u>: What is max size $r_2^<(N^2)$ of $S \subseteq [N]^2$ s.t.: $S$ does not contain a 3D corner?

3D corner:




(2) <u>Corners Coloring Problem</u>: What is min $c_2^<(N^2)$ st $[N]^2$ can be partitioned into $c_2^<(N)$ subsets s.t. no subset contains a 3D corner?

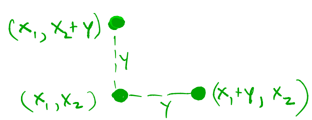
3D corner:




Again it is known that $c_2^<(N^2) \approx N / r_2^<(N^2)$

# Some Equivalences between Additive Comb's Problems + NOF CC Problems

[we will do case of $k=3$ but equivalences hold $\forall k \geq 3$]

| Additive Combinatorics Problem | Equivalent Comm. Complexity Problem |
|---|---|
| (1) 3-AP (Coloring) Problem   | (1') NIH Promise EQ, $k=3$<br>Player 1 : $x$<br>Player 2 : $y$<br>Player 3 : $z$<br>Promise: $x, y, z$ is a 3AP<br>Decide if $x=y=z$ |
| (2) Corners (Coloring) Problem   | (2') NOF ExactlyN $k=3$<br>Player 1 : $y, z$<br>Player 2 : $x, z$<br>Player 3 : $x, y$<br>Decide if $x+y+z=N$ |

**Also** : $CC(2') \leq CC(1')$ (So LBs for (2') $\Rightarrow$ LBs for (1')<br>So LBs for (2) $\Rightarrow$ LBs for (1))

# State of the Art: 3AP-Free sets

**Behrend** : 1946

$\exists$ a 3AP free subset of $[N]$ of size $N/\exp(2.25\sqrt{\log N})$

ie $r_3(N) \geq N/\exp(2.25\sqrt{\log N})$

$\therefore\ c_3(N) \leq \exp(2.25\sqrt{\log N})$

equivalently $\exists$ deterministic NIH Protocol for Promise EQ $k=3$ of cost $\log(c_3(N)) = O(\sqrt{\log N}) = O(n^{1/2})$

**Kelley, Meka** 2023

(Huge exponential improvement over previous results)

$r_3(N) \leq \dfrac{N}{\exp(c \log N^{1/12})}$

$\therefore\ c_3(N) \geq \exp(c \log N^{1/12})$

equivalently any det. NIH protocol for Promise EQ $k=3$ requires cost $\Omega(n^{1/12})$

# State of the Art: Corner free sets

Behrend: 1946

$\exists$ a corner-free subset of $[N^2]$ of size $N/\exp(c\sqrt{\log N})$

ie $r_3(N) \geq N/\exp(c\sqrt{\log N})$

$\therefore c_3(N) \leq \exp(\sqrt{\log N})$ $\leftarrow$ equivalently there is a deterministic NOF protocol for ExactlyN, $k=3$ of cost $O(n^{1/2})$

Best improvement to constant $c$: $c \sim 1.8$ [Linial, Shraibman '21] [Green '21]

Jaber, Liu, Lovett, Ostuni, Sawhney 2025

$r_2^<(N^2) \leq N^2/\exp(c \cdot (\log N)^{1/200})$

$\therefore c_2^<(N^2) \geq \Omega(n^{1/200})$ $\leftarrow$ equivalently any NOF protocol for ExactlyN $k=3$ has cost $\Omega(n^{1/200})$

## ① 3-AP Coloring Problem $\approx$ ①' NIH Promise EQ, $k=3$


Player 1 : $x$
Player 2 : $y$
Player 3 : $z$

Promise: $x, y, z$ is a 3AP

Decide if $x = y = z$

$\Rightarrow$ : Let $S_1, \ldots, S_{c_3(N)}$ be a partition of $[N]$ into $c_3(N)$ subsets, all $S_i$ 3AP free.

Protocol for NIH promise EQ (on input $(x, y, z)$):

P1 sends $i \in c_3(N)$ s.t. $x \in S_i$

P2 : sends 1 iff $y \in S_i$

P3 : send 1 iff $z \in S_i$

output 1 iff P2, P3 both send 1.

Since $(x, y, z)$ is a 3AP, they are in same set $S_i$ iff $x = y = z$ (because each $S_i$ contains NO nontrivial 3AP)

Complexity of protocol: $\log(c_3(N)) + 2$

① 3-AP Coloring Problem $\approx$ ①' NIH Promise EQ, $k=3$


Player 1 : $x$
Player 2 : $y$
Player 3 : $z$

Promise: $x, y, z$ is a 3AP

Decide if $x = y = z$

$\Leftarrow$ : Let $\pi$ be a NIH protocol for Promise EQ.

Using $\pi$, we define a coloring (= partition) of $[N]$:

$$\text{Color}(x) \stackrel{d}{=} \text{Transcript of } \pi \text{ on } (x, x, x)$$

We claim $\forall S \subseteq [N]$ s.t. all elements in $S$ have same color (transcript), $S$ contains NO 3AP:

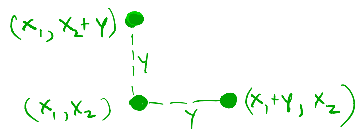
Assume otherwise, so $x, \overbrace{x+\delta}^{y}, \overbrace{x+2\delta}^{z}$ ($\delta > 0$) have same transcript

Then since protocol is NIH, $\pi(x, y, z) = \pi(x, x, x) = \pi(y, y, y) = \pi(z, z, z)$

by rectangular prop of $\pi$

But then protocol is incorrect since $(x, x, x), (y, y, y), (z, z, z)$ are 1-inputs but $(x, y, z)$ is a 0-input

## ② Corners Coloring Problem $\approx$ ②' NOF ExactlyN $K=3$




Player 1: $y, z$
Player 2: $x, z$
Player 3: $x, y$

Decide if $x+y+z=N$

$\Rightarrow$: Assume $[N]^2$ has a partition (coloring) into $c_2^{<}(N)$ corner-free subsets.

Consider the inputs: $\{(x, y), (x, y+d), (x+d, y)\}$, $d = N-x-y-z$

They have same color iff $d=0$ iff $N = x+y+z$

### Protocol For Exactly N on $(x, y, z)$:

Player 3 (sees $x, y$): sends color$(x, y)$

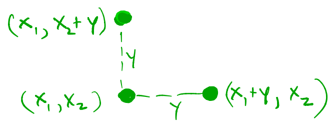
Player 2 (sees $x, z$): sends 1 iff color$(x, y) =$ color$(x, \overbrace{y+d}^{N-x-z})$

Player 1 (sees $y, z$): sends 1 iff color$(x, y) =$ color$(\underbrace{x+d}_{N-y-z}, y)$

output 1 iff Players 2, 3 both send 1.

Complexity of protocol $= \log(c_2^{<}(N)) + 2$

## ② Corners (coloring) Problem $\approx$ ②' NOF ExactlyN $k=3$




Player 1: $y, z$
Player 2: $x, z$
Player 3: $x, y$

Decide if $x+y+z=N$

$|x|=|y|=|z|=n=\log N$

$\Leftarrow$: Let $\Pi$ be a $c$-bit protocol for ExactlyN.
Using $\Pi$ we give a coloring of $[N]^2$ s.t. each color class/partition is corner-free.

$\text{Color}(x,y) :=$ Transcript of $\Pi$ on $(x, y, N-x-y)$ $\leftarrow$ # colors $= 2^c$

Claim: Each color class is corner free.

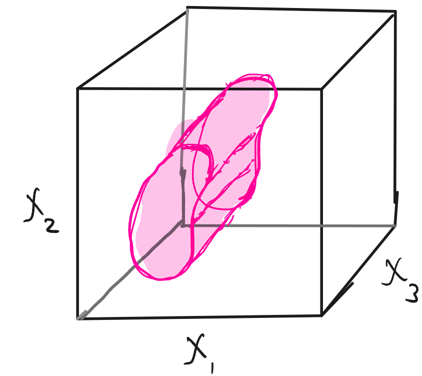
If not, then $\exists x, y, d>0$ s.t. $(x,y), (x+d, y), (x, y+d)$ have same color

So $(x, y, N-x-y), (x+d, y, N-x-y-d), (x, y+d, N-x-y-d)$ have same transcript

$\therefore (x, y, N-x-y-d)$ has same transcript also (By cylinder intersection property of $\Pi$)

# NOF Lower Bounds via Discrepancy [BNS]

$k=3$:




<u>Defn.</u> Let $X = X_1 \times X_2 \times \cdots \times X_k$ (input space)

A cylinder $C_i$ in $i^{th}$ coordinate is a subset of $X$ that doesn't depend on $i^{th}$ coordinate

i.e. $(x_1, .., x_i, .. x_k) \in C_i \Rightarrow \forall x_i' \in X: (x_1, .., x_i', .. x_k) \in C_i$

<u>Defn</u> A <u>cylinder intersection</u> is a subset $C \subseteq X$ s.t. $C = C_1 \cap C_2 \cap \cdots \cap C_k$, where $C_i$ = cylinder intersect in $i^{th}$ coord.

<u>Claim</u> Let $\Pi$ be a cost $c$ NOF protocol. then $\Pi$ induces a partition of $X$ into $2^c$ cylinder intersections

Analog of 2-party partition into <u>rectangles</u>

$\therefore$ If $\Pi$ is a NOF protocol for $f : X_1 \times X_2 \times \dots \times X_k \to \{0,1\}$ then $\Pi$ induces a partition of $\mathcal{X}$ into $2^c$ f-monochromatic cylinder intersections

<u>Defn</u> (Discrepancy for cylinder Intersections)

Let $f : X \to \{\pm 1\}$, and cylinder intersection $C : X \to \{0,1\}$

$$\mathrm{Disc}_\mu(f, C) = \left| \mathop{\mathbb{E}}_{(x_1 \dots x_k) \sim \mu} [f(x_1 \dots x_k)\, C(x_1 \dots x_k)] \right|$$

$$\mathrm{Disc}_\mu(f) = \max_C \mathrm{Disc}_\mu(f, C)$$

same as Defn of $\mathrm{Disc}(f, R)$ but now over cylinder intersections

<u>Lemma</u> $D_k^{\varepsilon, \mu}(f) \geq \log\left(\frac{1 - 2\varepsilon}{\mathrm{Disc}_\mu(f)}\right)$

Distrib. cc of k-player deterministic NOF of f

Proof same as for Disc over Rectangles

**Theorem** Let $\mu$ = unit distribution on $X$.

$$D_k^{\frac{1}{3},\mu}(g\,IP_n) = \Omega\left(\frac{n}{4^k}\right)$$

**Main Lemma** $\mathrm{Disc}_\mu(g\,IP_n) \leq \exp(-n/4^k)$

Proof is very similar to our "BNS" proof of 2-player lower bound for $IP_n$

But now we apply Cauchy Swartz (Jensen's Ineq) $k-1$ times. Each time we lose a factor of $\sim 4$ in the LB.