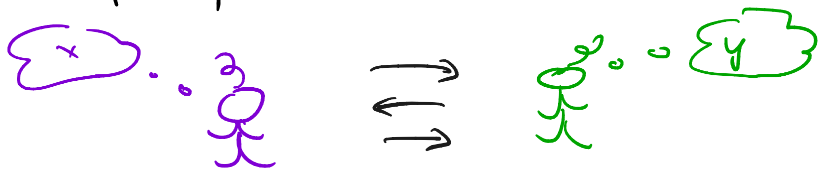
Last Class

1. 2-party basic model (deterministic)



$$P^{cc}(f) = \min_{\Pi \text{ for } f} \ \max_{\substack{(x,y) \\ |x|=|y|=n}} \ \# \text{ bits sent on input } (x,y)$$

P^{cc} = class of all functions $f: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ st $P^{cc}(f) = (\log n)^{O(1)}$

2. Matrix view of protocol (partitions M_f into disjoint monochrom. subrectangles)

Logrank conjecture

LB for EQ_n

Today

1. Deterministic Protocols & partition number vs det. cc partition
   balancing protocols

2. Randomized cc Protocols : one-sided (RP^{cc}) + 2-sided

2a. Error ε can be amplified with little cost

Example EQ_n : has $O(1)$ cost public coin RP^{cc} protocol

Deterministic CC

We saw that any det. protocol for $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ gives a partition of M_f into monochrom subrectangles

Let the partition # of M_f = min size (# of rectangles) over all partitions of M_f into monochrom. subrectangles

Q: Does $\forall M_f$ partition # of M_f $= 2^{\log(cc(f))}$?

Deterministic CC

Q $\forall f$ Is $\log(\text{partition } \# \text{ of } f) = O(CC(f))$?

Theorem [Yannakakis alg 1]

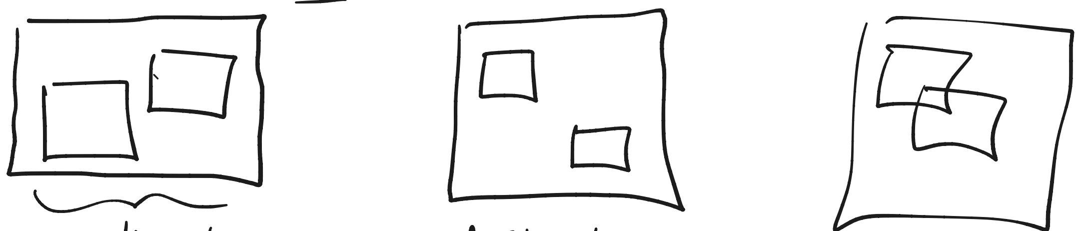
If f has size $\leq 2^c$ partition number then $CC(f) \leq c^2$

Thm 2 [Goos - P - Watson]

The above thm is tight !

Yannakakis Alg 1

Fact: any 2 disjoint rectangles cannot intersect in rows <u>and</u> in columns



intersect in rows

don't intersect in rows or columns

intersect in <u>both</u>

Let $R = R_0 \cup R_1$ be a partition of M_f

R_0 = 0-rectangles, R_1 = 1-rectangles

Let $(x,y) \in f^{-1}(0)$.

Then let $R_{x,y} \in R_0$ be unique 0-rectangle containing (x,y)

Then every $R \in R_1$ has to intersect $R_{x,y}$ either in rows or cols.

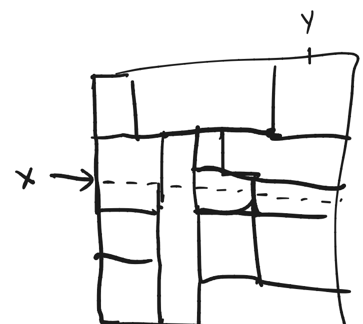
$\therefore$ either (1) $\leq$ half of rectangles in R_1 interset $R_{x,y}$ in rows
or (2) $\leq$ half of rectangles in R_1 " $R_{x,y}$ in cols

$R = A \times B \in R_0$ is row-good if $\leq$ half of rect's in R_1 intersect A and $x \in A$
" col-good if $\leq$ " " " " R_1 " B and $y \in B$

Alice on x: Knows all $R \in R_0$ that contain row x

so she can compute whether or Not there is a row-good $R \in R_0$

Similarly Bob can compute whether or not there is a col-good $R \in R_0$



$R = A \times B \in R_0$ is row-good if $\leq$ half of rect's in R_1 intersect A and $x \in A$
" col-good if $\leq$ " " " " R_1 " B and $y \in B$

Alice on x: Knows all $R \in R_0$ that contain row x so she can compute whether or not there is a row-good $R \in R_0$

Similarly Bob on y: can compute whether or not there is a col-good $R \in R_0$

Protocol

→ Alice/Bob send name of rectangle that is row-good or col-good for (x,y).

If no row or col-good rectangle exists → players output 1

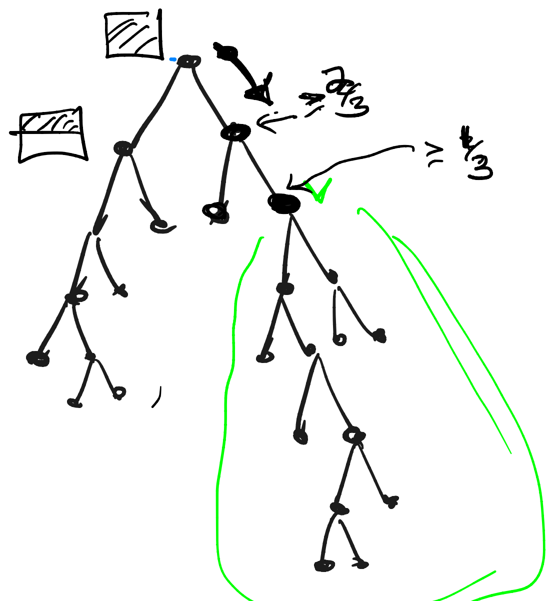
OW one exists, this removes $\geq \frac{1}{2}$ of rect's in R_1 so repeat using R_0, $R_1 = R_1 - \{\text{discarded rect's in } R_1\}$

Complexity since $|R_1| \leq 2^c$, # rds of protocol $\leq c$, + each one sends $O(c)$ bits

BALANCING PROTOCOLS

Theorem If f has a deterministic protocol Π with ℓ leaves, then f has a det protocol of height $O(\log \ell)$

$\frac{1}{3}$-$\frac{2}{3}$ Lemma Any binary tree T with $\ell > 1$ leaves contains a vertex v st T_v has between $\frac{\ell}{3}$ and $\frac{2\ell}{3}$ leaves



BALANCING PROTOCOLS

$\boxed{\text{Theorem}}$ If f has a deterministic protocol Π with ℓ leaves, then f has a det protocol of height $O(\log \ell)$

$\boxed{\tfrac{1}{3}\text{-}\tfrac{2}{3}\text{ Lemma}}$ Any binary tree T with $\ell > 1$ leaves contains a vertex v st T_v has between $\frac{\ell}{3}$ and $\frac{2\ell}{3}$ leaves

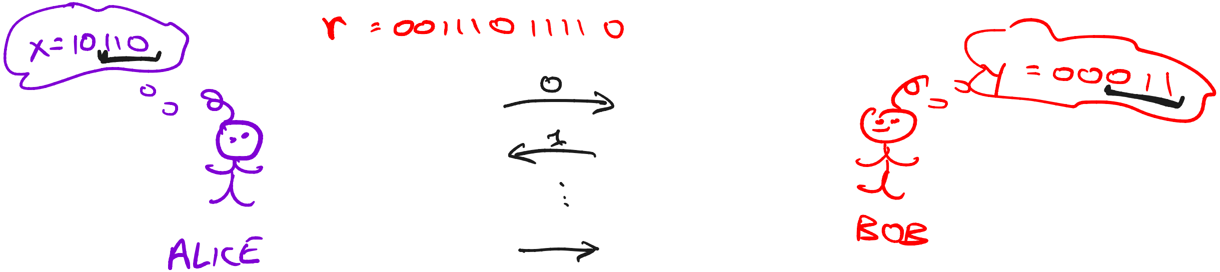
given Π, with ℓ leaves:

1. Players (no communication) find $\frac{1}{3}$-$\frac{2}{3}$ vertex v
2. Alice sends one bit – 1 iff $x \in R_v$
   Bob " " " – 1 iff $y \in R_v$
3. If alice + Bob both send 1 then recurse on T_v
   ow delete T_v from T + recurse on $T - T_v$

at each round # leaves in current tree shrinks by at least $\frac{2}{3}$ factor

so # bits $\leq 2 \cdot \log_{2/3} \ell = O(\log \ell)$

Randomized COMMUNICATION COMPLEXITY (public coin)



Alice/Bob have shared access to a random string $r \in \{0,1\}^R$

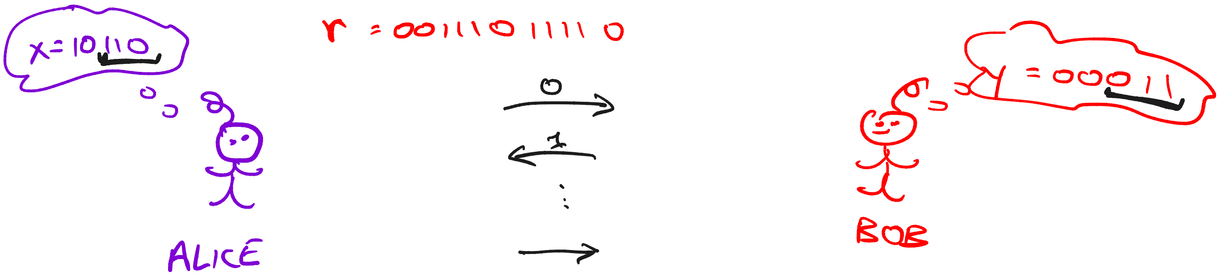
For each $r \in \{0,1\}^R$, have a deterministic protocol Π_r (depends on r)

So randomized protocol Π is a distribution over deterministic protocols $\{\Pi_r\}_{r \in \{0,1\}^R}$

Π computes $f : X \times Y \to \{0,1\}$ with prob $\geq 1-\varepsilon$ if:

$\forall (x,y) \in f^{-1}(1)$ $\Pr_r[\Pi_r(x,y) = f(x,y)] \geq 1-\varepsilon$

Randomized COMMUNICATION COMPLEXITY (public coin)



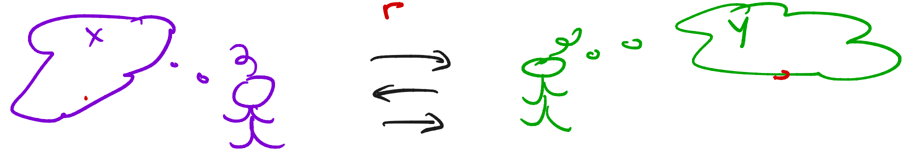
Π computes $f : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ with prob $\geq 1-\varepsilon$ if:

$$\forall (x,y) \in f^{-1}(1) \quad \Pr_r\left[\Pi_r(x,y) = f(x,y)\right] \geq 1-\varepsilon$$

CC of (randomized) $\Pi := \max_r \left(CC(\Pi_r) \right)$ ← we'll see that wlog $|r| \leq \text{polylog}\, n$

Randomized CC

(Public Coin Model)



BPP^{cc}

Π computes f_n with error ε if: $\forall (x,y)$ $|x|=|y|=n$

$$\Pr_{r_A, r_B}\left[\Pi(x, y, r) = f_n(x,y)\right] \geq 1-\varepsilon$$

Default $\varepsilon = 1/3$

$f \in BPP^{cc}_{\varepsilon}(f)$ if randomized CC of f $= O(\text{polylog}\, n)$

RP^{cc}

Π computes f with 1-sided error ε if $\forall (x,y)$ $|x|=|y|=n$

$$f(x,y)=1 \Rightarrow \Pr_r\left[\Pi(x,y,r) = f(x,y)\right] = 1$$

$$f(x,y)=0 \Rightarrow \Pr_r\left[\Pi(x,y,r) = f(x,y)\right] \geq 1-\varepsilon$$

$f \in RP^{cc}(f)$ if randomized cc of $f = O(\text{polylog}\, n)$

ZPP^{cc}

error $\varepsilon = 0$.

$$f \in ZPP^{cc}: \min_{\Pi \text{ for } f} \max_{(x,y)} \mathbb{E}_r\left[\#\text{ bits sent on } (x,y)\right]$$

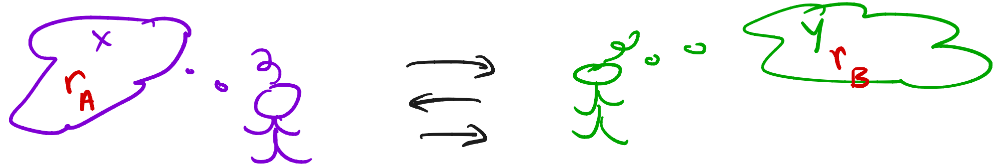
Sometimes we'll also use

$P^{cc}(f)$, $BPP^{cc}(f)$, $RP^{cc}(f)$, etc

to denote the comm complexity of f in that model.

Randomized CC

(Private Coin Model)



$\boxed{BPP^{cc}}$ Π computes f with error ε if: $\forall (x,y)$ $|x| = |y| = n$

$$\Pr_{r_A, r_B}\left[\Pi(x, r_A; y, r_B) = f(x,y)\right] \geq 1 - \varepsilon$$

Default $\varepsilon = 1/3$

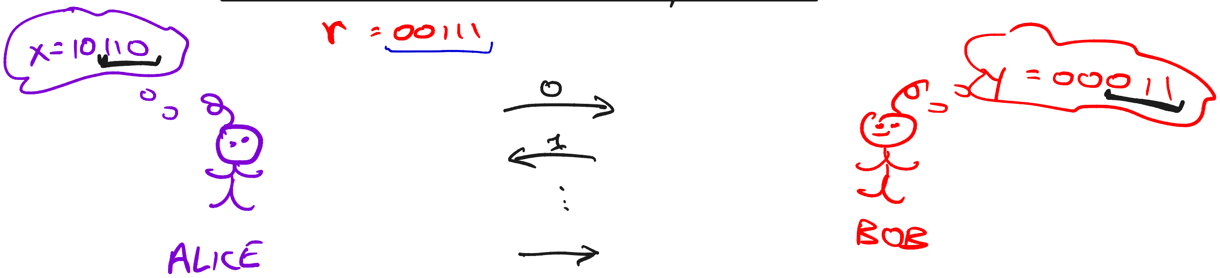
$\boxed{RP^{cc}}$ Π computes f with 1-sided error ε if $\forall (x,y)$

$$f(x,y) = 1 \Rightarrow \Pr_{r_A, r_B}\left[\Pi(x, r_A; y, r_B) = f(x,y)\right] = 1$$

$$f(x,y) = 0 \Rightarrow \Pr_{r_A R_B}\left[\Pi(x, r_A; y, r_B) = f(x,y)\right] \geq 1 - \varepsilon$$

$\boxed{ZPP^{cc}}$ error $\varepsilon = 0$.

Example: EQUALITY



We saw last lecture: deterministic cc of $EQ_n = \Omega(n)$

RP^{cc} EQ protocol ($\varepsilon = \frac{1}{2}$)

- View 1st n bits of r as selecting a subset of $1 \ldots n$.
- Alice sends parity of $x|_r$
- Bob sends parity of $y|_r$
- Accept (output 1) iff parities are the same

  o/w output 0

Repeat $O(\log n)$ times to get $O(\log n)$-cost protocol with $\varepsilon \sim \frac{1}{n}$

Amplification of error in Randomized Protocols

RP^{cc} protocols:

Say Π is an RP^{cc} protocol for f with error ε

To decrease $\varepsilon \to \varepsilon^k$:

Repeat Π k times using $r_1, .. r_k$

Output 1 if $\forall$ k runs, Π output 1

Else output 0

If $f(x,y) = 1$ $\quad Pr(\Pi_{r_1} = 1 \wedge \Pi_{r_2} = 1 \wedge \ldots \wedge \Pi_{r_k} = 1) = 1$

If $f(x,y) = 0$ $\quad Pr(\Pi(x,y) = 1) \wedge Pr(\Pi_{r_1}(x,y) = 1) \wedge .. \wedge Pr_{\Pi_k}(x,y) = 1)$

$= \varepsilon^k$

$\therefore Pr(\Pi(x,y) = f(x,y)) = 1 - \varepsilon^k$

Amplification of error in Randomized Protocols

BPP^{ϵ} protocols:

Say Π is an RP^{cc} protocol for f with error ε

To decrease $\varepsilon \longrightarrow \varepsilon^{k}$:

Repeat Π k times using $r_1, \ldots r_k$

Output majority answer