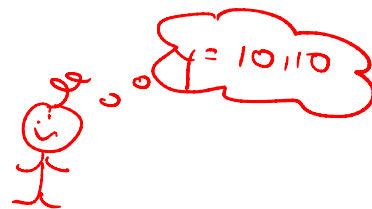
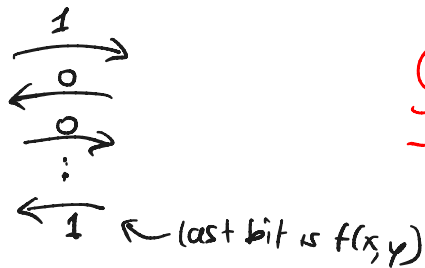


CS 6998:
COMMUNICATION COMPLEXITY & APPLICATIONS



CS 6998:
COMMUNICATION COMPLEXITY & APPLICATIONS

Toniann Pitassi toni@cs.columbia.edu

Course webpage: www.cs.columbia.edu/~toni
click on teaching Link

Lectures: Wed 1:10 - 4 (we will almost always end by 3:15)

Office hrs: by appointment

All course materials provided on webpage, including lecture notes

Optional Textbooks:

Nisan-Kushilevitz	Communication Complexity
Rao-Yehudayoff	comm comp + Applications

] Both have prelim.
draft book ONLINE

Course Outline/Evaluation (see webpage)

- 1-2 assignments
- class participation
- presentations

First part of class : I will lecture on fundamentals of communication complexity (~8-10 weeks)

Second part of class : Presentations

* We will have ~ 2-week study period (No lectures) to prepare presentations

Jan 28

Feb 4, 11, 18, 25

Mar 4, 11

Mar 18 : No lecture (spring break)

Mar 25 : presentations or Toni lecture

Apr 1 : Presentations or Toni Lecture

Apr 8 : No Lecture (prepare presentations)

Apr 15, 21 : Presentations (Last class Apr 21)

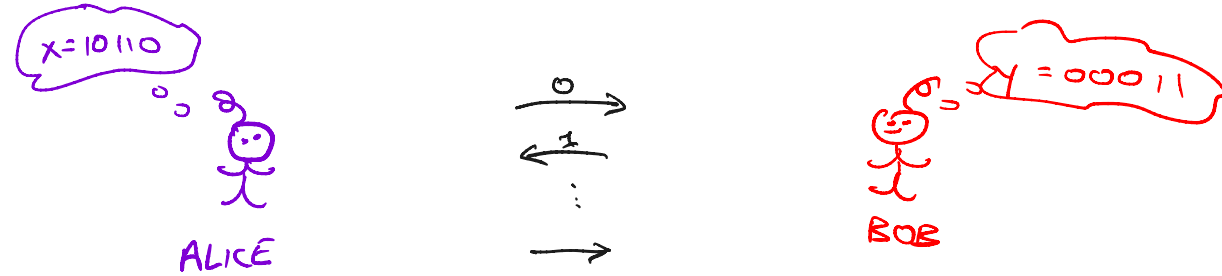
Presentations

List of papers/topics

Each presentation has: presenter(s)
 supporter(s)

Teams will meet with me before to go over presentation

COMMUNICATION COMPLEXITY



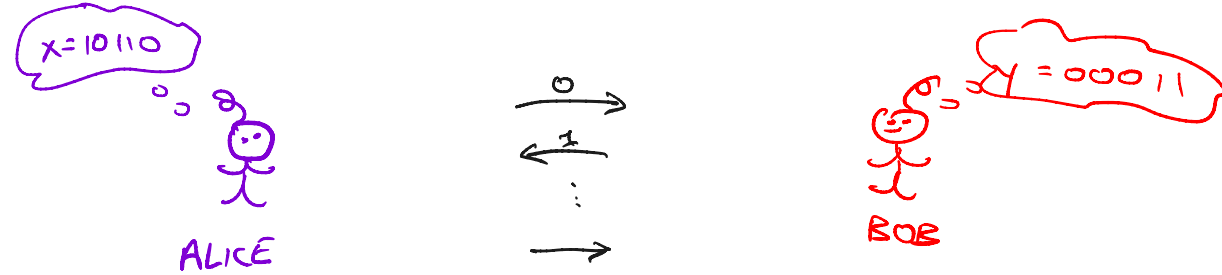
Alice & Bob have private inputs x and y respectively

* Typically $|x| = |y| = n$

They want to compute some joint function (or search problem) $F(x, y)$ by sending bits back & forth (bits can be arbitrary, not nec. bits of x or y)

* Model is nonuniform: separate CC protocol for each n

COMMUNICATION COMPLEXITY

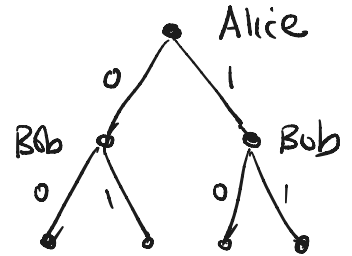


Example 1 $\text{Parity}(x, y) = \text{parity of number of 1's in combined string } xy$

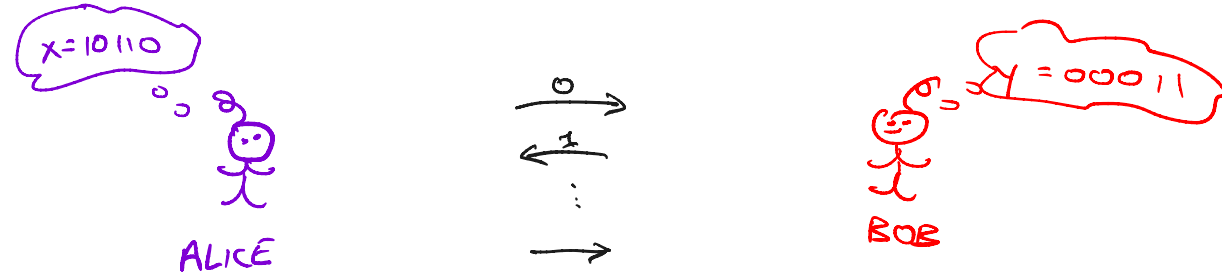
2 bit protocol:

Alice sends parity of # 1's of x

Then Bob sends parity of x , plus bit sent by Alice



COMMUNICATION COMPLEXITY

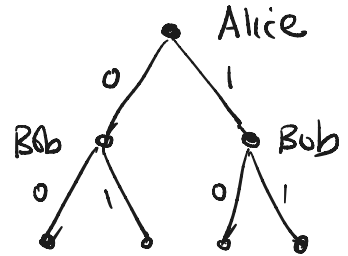


Example 1 $\text{Parity}(x, y)$ = parity of number of 1's in combined string xy

$CC(\Pi_n) \stackrel{d}{=} \max \# \text{ bits sent over all } x, y \text{ } |x|=|y|=n$

$CC(F) \stackrel{d}{=} \min_{\{\Pi_n\}} CC(\Pi_n)$

so $CC(\text{parity}) = 2 = O(1) \leftarrow \text{independent of } n!$
so parity is very easy.

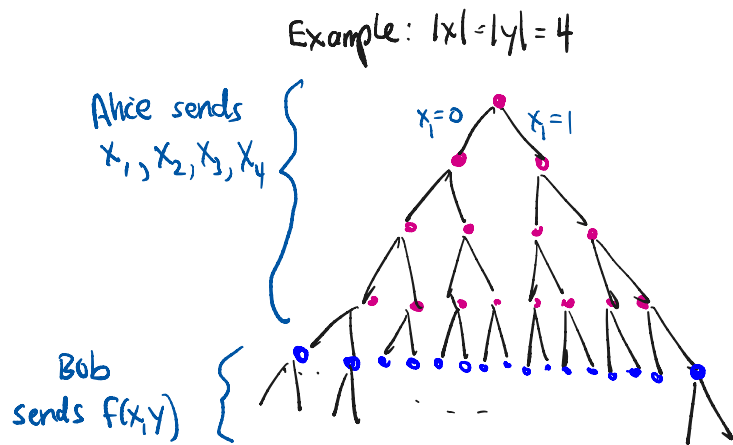


Observation: $\forall F: \underbrace{\{0,1\}^n}_x \times \underbrace{\{0,1\}^n}_y \rightarrow \{0,1\}$, $cc(F) \leq n+1$

Protocol:

Alice sends x to Bob in n rounds

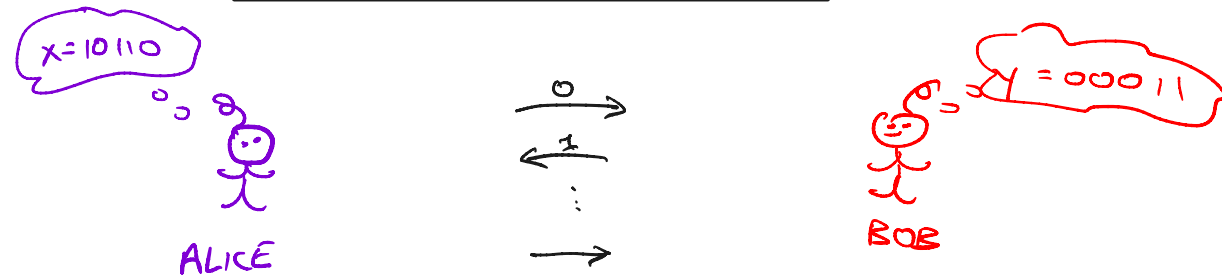
Bob computes $F(x,y)$ and sends answer



$\therefore$ we consider F "easy" for deterministic CC
 $\text{if } cc(F) = \text{polylog}(n)$

$$P^{cc} \stackrel{d}{=} \left\{ \left\{ F_n: \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\} \right\}_n \mid \Pi(F_n) = O(\text{polylog } n) \right\}$$

COMMUNICATION COMPLEXITY



Example 2 $EQ(x, y) = 1 \text{ iff } x = y$

2-party deterministic protocols (More formal Defn)

2 sets of inputs X, Y (usually $X = \{0,1\}^n = Y$)

A protocol π specified by a rooted tree, fanout-2

- each vertex associated with one player (Alice or Bob)

- If vertex v assoc. with Alice, then

Alice computes a fxn $f_v : X \rightarrow \{0,1\}$ at v

- If v assoc. with Bob, then Bob computes

$f_v : Y \rightarrow \{0,1\}$

- Alice
- Bob

- Alice
- Bob

- each vertex associated with one player (Alice or Bob)
- If vertex v assoc. with Alice, then Alice computes a f.x.N $f_v: X \rightarrow \{0,1\}$ at v
- If v assoc. with Bob, then Bob computes $f_v: Y \rightarrow \{0,1\}$

On input $(x, y) \in X \times Y$:

Players start at root of protocol tree

If at vertex v : player assoc. with v sends $f_v(x)$ to other player
+ both players move to child v' indicated by $f_v(x) \in \{0, 1\}$.

Repeat until players reach a leaf vertex, v_ℓ , and output value of this leaf

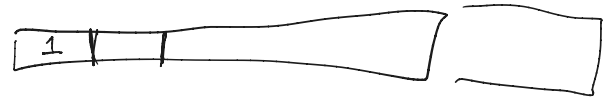
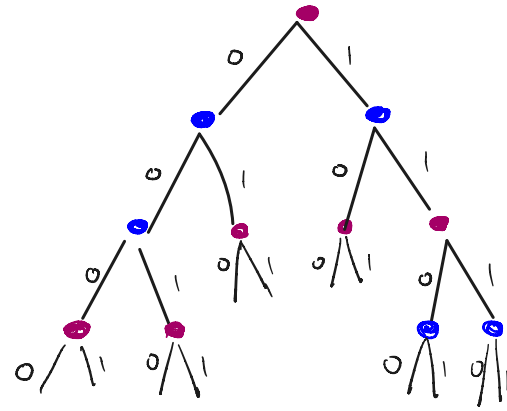
Π computes $g: X \times Y \rightarrow \{0,1\}$ if

$$\forall (x,y) \in X \times Y \quad \Pi(x,y) = g(x,y)$$

The comm. complexity of Π , $\|\Pi\|$, is
the depth of protocol tree

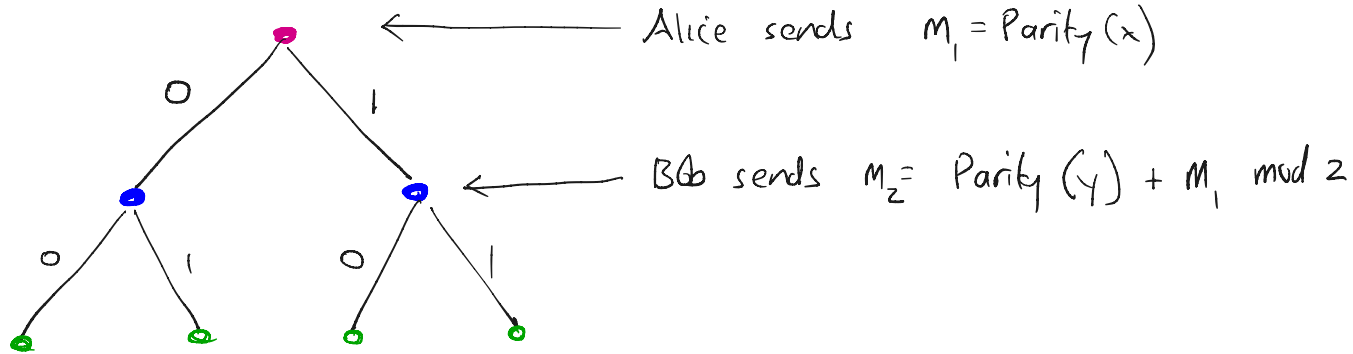
The number of rounds of Π is max
of alternations (red/blue) over
all paths of protocol tree.

• Alice
• Bob



if $y_{x_1}^1 = 1$ output a
else if $y_{x_2}^2 = 1$ output b
else output 0

Example 1 Protocol for Parity (x, y)



ex: $x = 00110$: $m_1 = 0$
 $y = 10110$: $m_2 = m_1 + \text{Parity}(y) \bmod 2$
 $= 0 + 1 \bmod 2$
 $= 1$

2 rounds

comm. complexity = 2 $\leftarrow$ for any input lengths $|x|, |y|$

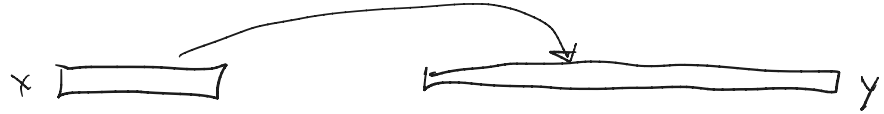
Another example: 2-Level Pointer jumping

Index function

Alice gets $x \in \{0,1\}^n$

Bob gets $y \in \{0,1\}^{2^n}$

output $y|_x$

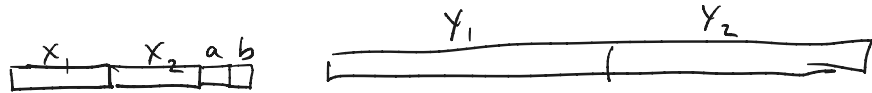


Pointer Jumping Function:

Alice gets $x^1, x^2 \in \{0,1\}^n$

$a, b \in \{0,1\}$

Bob gets $y^1, y^2 \in \{0,1\}^{2^n}$



If $y_1|_{x_1} = 1$ output a

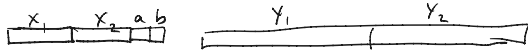
else if $y_2|_{x_2} = 1$ output b

else output 0

Pointer Jumping Function:

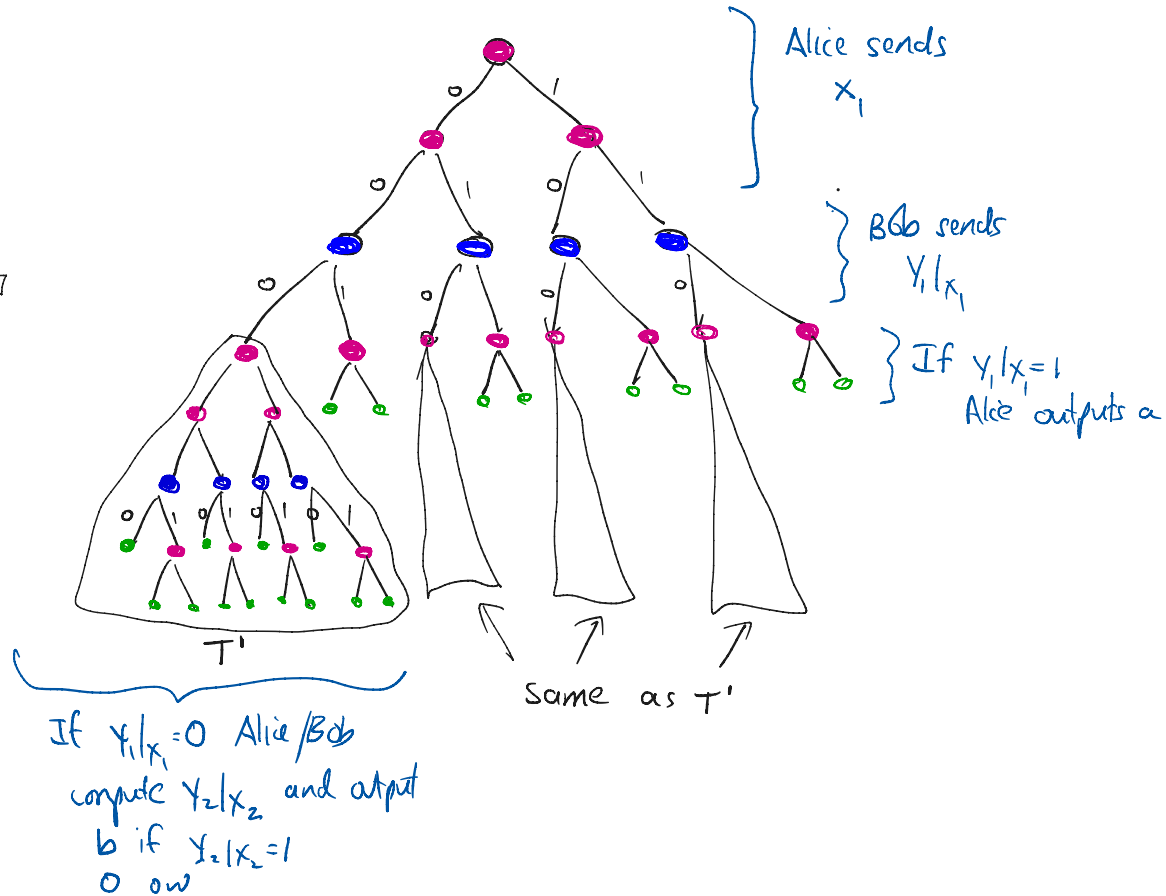
Alice gets $x^1, x^2 \in \{0,1\}^n$
 $a, b \in \{0,1\}$

Bob gets $y^1, y^2 \in \{0,1\}^{2^n}$



If $y_1/x_1 = 1$ output a
else if $y_2/x_2 = 1$ output b
else output 0

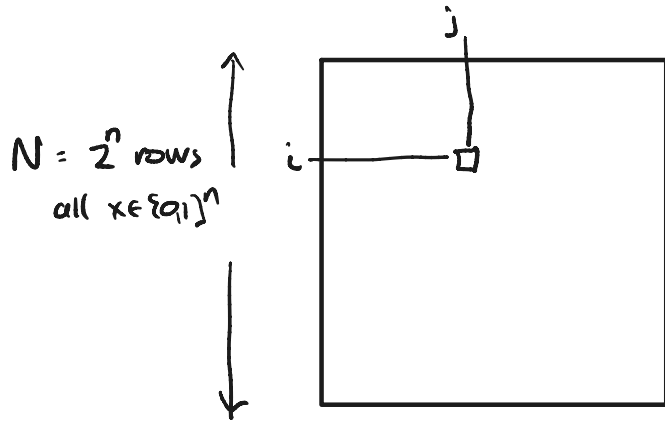
Protocol ($n=2$)



Equivalent Matrix view of Deterministic CC protocol, Π :

$$\text{Let } f: \underbrace{\{0,1\}^n}_x \times \underbrace{\{0,1\}^n}_y \rightarrow \{0,1\}$$

the comm. matrix M_f associated with f :

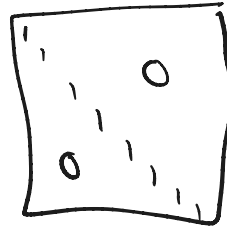


$$M_{ij} = f(c, j)$$

$$\leftarrow 2^n = N \text{ cols} \rightarrow$$

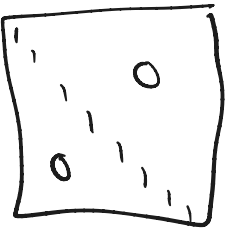
all $y \in \{0,1\}^n$

$$M_{\text{EQ}} = \text{Identity matrix}$$



Examples

1. Equality : $EQ(x, y) = 1$ iff $x = y$

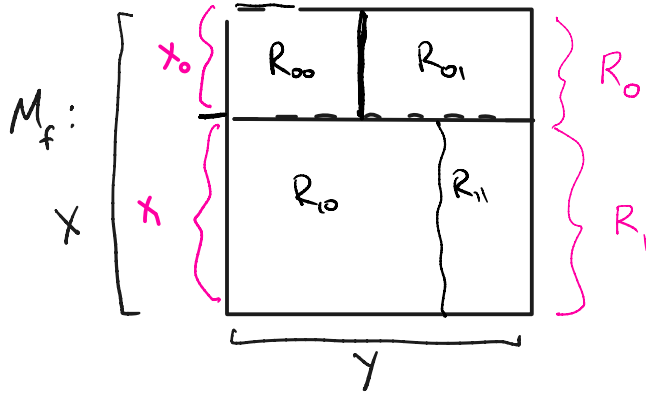
M_{EQ} :  ← Identity matrix

2. $\text{Parity}(x, y) = 1$ iff Number of 1's in $x =_{\text{mod } 2} \# \text{ 1's in } y$

M
PARITY

	000	001	010	011	100	101	110	111
000	1	0	0	1	0	1	0	1
001	0	1	1	0	1	0	0	1
010	0	1	1	0	1	0	0	1
011	1	0	0	1	0	1	1	0
100	0	1	1	0	1	0	0	1
101								
110								
111	0	1	1	0	1	0	0	1

Matrix View of Protocol Π for $f(x, y)$

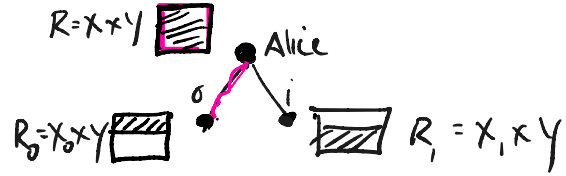


Let $X_0 = \{x \in X \mid \text{Alice's 1st bit is } 0\}$
 $X_1 = \{x \in X \mid \text{Alice's 1st bit is } 1\}$

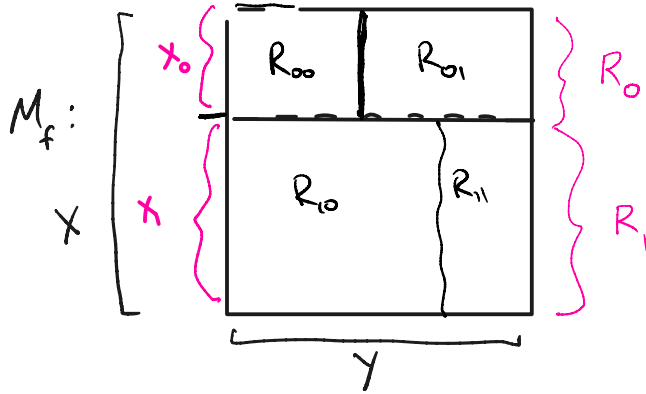
This partitions M_f into 2 disjoint subrectangles: $R_0 = X_0 \times Y$
 $R_1 = X_1 \times Y$

Defn A subrectangle R of M has the form $X' \times Y'$ where $X' \subseteq X$, $Y' \subseteq Y$

Say Alice sends 1st bit :



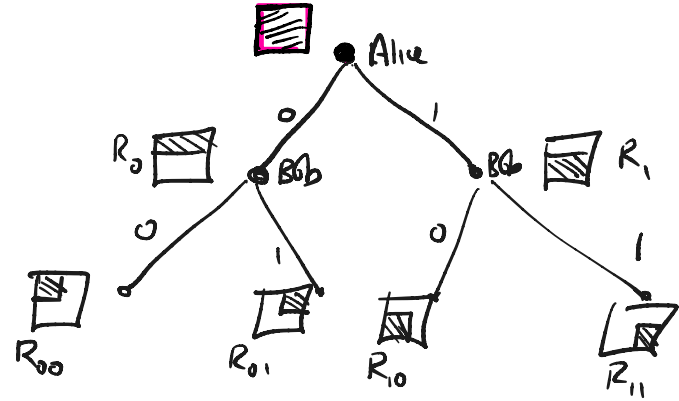
Matrix View of Protocol Π for $f(x,y)$



Let $X_0 = \{x \in X \mid \text{Alice's 1st bit is } 0\}$
 $X_1 = \{x \in X \mid \text{Alice's 1st bit is } 1\}$

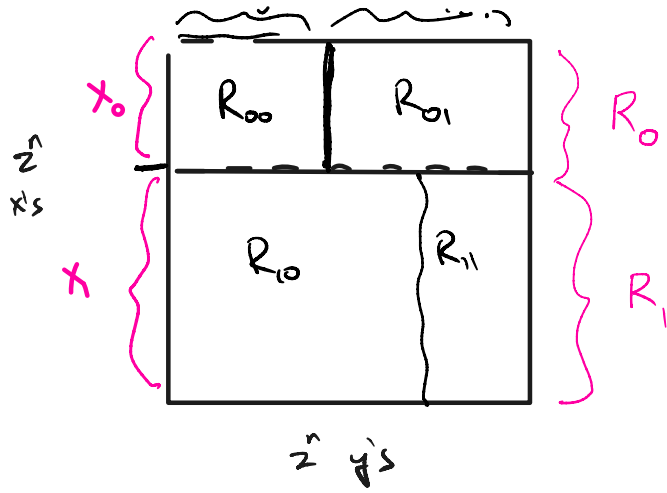
Next bit sent further
 partitions M into 4 disjoint
 Rectangles : R_{00}
 R_{01}
 R_{10}
 R_{11}

Say Bob sends 2nd bit :

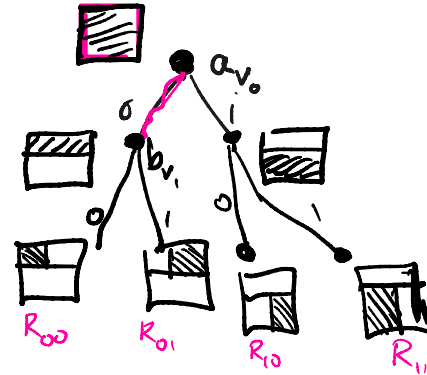


R_{00} = inputs in $X \times Y$
 corresponding to transcript 00

Matrix View of Protocol Π for $f(x,y)$

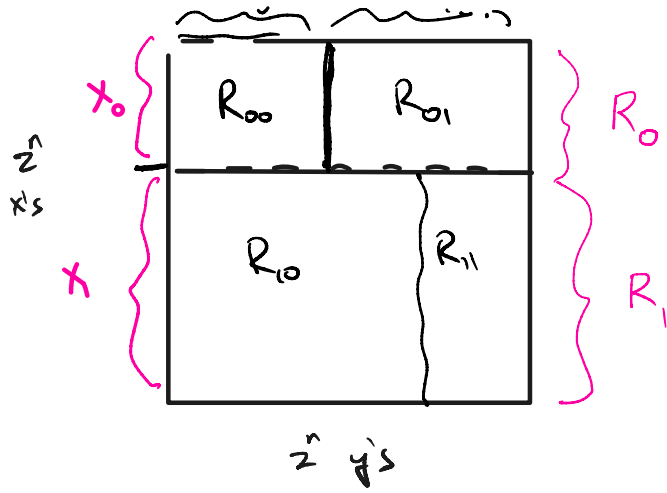


Say Alice sends 1st bit

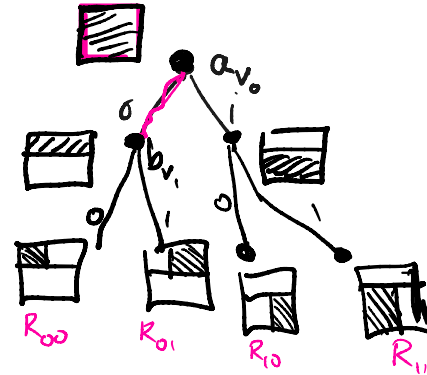


At each leaf of protocol tree,
 output is last bit sent, which
 should equal $f(x,y) \forall (x,y) \in R_\ell$
 $\therefore$ The subrectangles at leaves are all
monochromatic

Matrix View of Protocol Π for $f(x, y)$



Say Alice sends 1st bit



R_{00} = inputs in $X \times Y$ corresponding to transcript 00

R_{01}

R_{10}

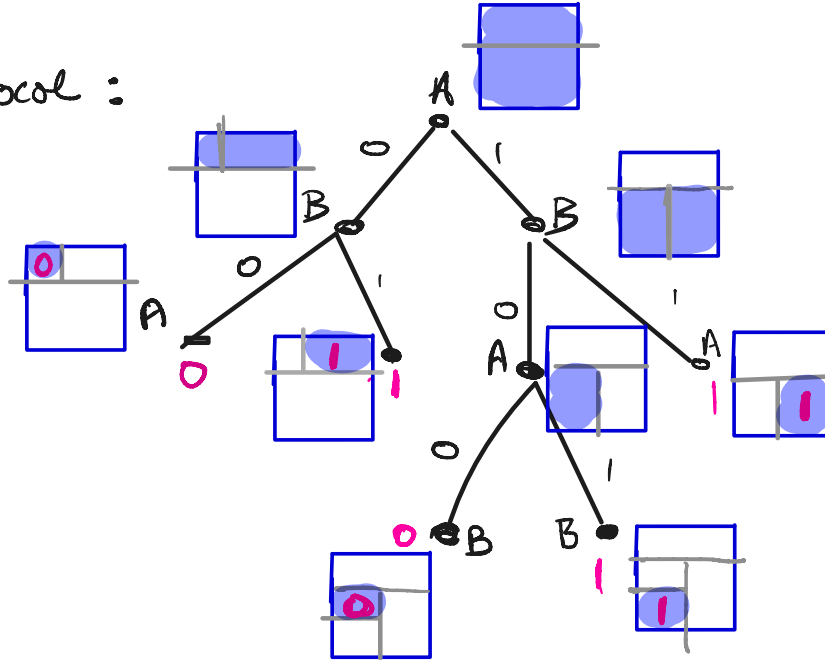
R_{11}

$\therefore$ A deterministic protocol Π partitions M_f into disjoint monochromatic subrectangles

Another visualization:

f , $M_f = \text{cc matrix for } f$

deterministic protocol:

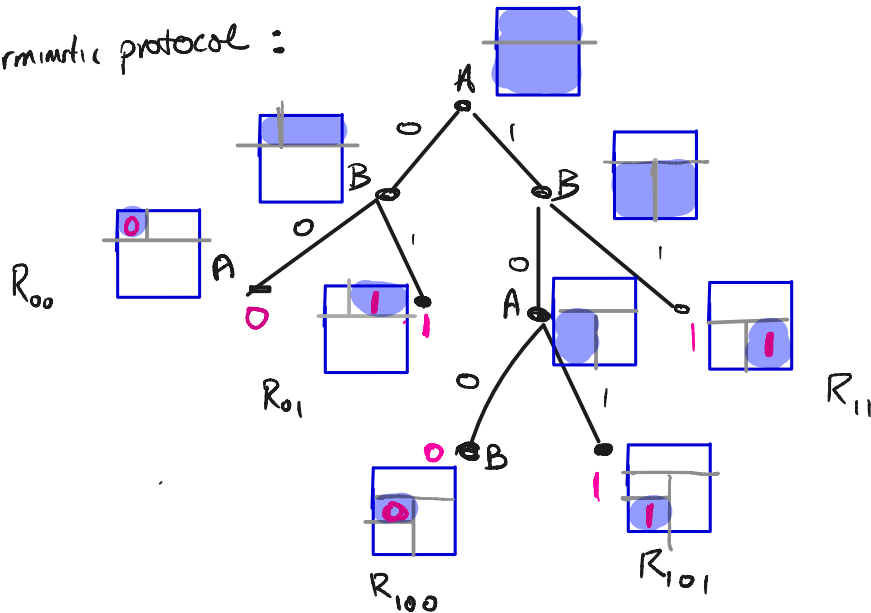


A = Alice speaks
B = Bob "

Last Class cont'd

f , $M_f =$ cc matrix for f

deterministic protocol:



$$\therefore M_f = \begin{matrix} \text{blue square} \end{matrix} = \begin{matrix} \text{blue square with red 0 in top-left} \end{matrix} + \begin{matrix} \text{blue square with red 1 in top-right} \end{matrix} + \begin{matrix} \text{blue square with red 0 in bottom-left} \end{matrix} + \begin{matrix} \text{blue square with red 1 in bottom-right} \end{matrix} + \begin{matrix} \text{blue square with red 1 in bottom-right} \end{matrix}$$

$R_{00} \quad R_{01} \quad R_{100} \quad R_{101} \quad R_{11}$

LogRank Conjecture in CC

By the previous argument, a deterministic CC protocol of cost d for $f: \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$ gives a partition of M_f into 2^d disjoint monochromatic subrectangles:

$$\text{That is } M = \sum_{\alpha \in \{0,1\}^d} R_\alpha \quad \alpha: \text{transcript of protocol}$$

Each R_α has rank ≤ 1 (since all-0 outside M and inside R_α either all 1 or all 0)

$$\therefore \text{Rank}(M) \leq \sum_{\alpha} \text{Rank}(R_\alpha) \leq 2^d$$

Log Rank Conjecture in CC

This shows: For any $f: \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$ the

$$\text{det cc of } f: \text{CC}(f) \text{ is } \geq \log_2(\text{rank}(M_f))$$

Log Rank conjecture: states that up to poly factors this is an equivalence

Log Rank Conjecture

$$\log_2(\text{Rank}(M_f)) \leq \text{CC}(f) \leq \log_2(\text{Rank}(M_f))^{O(1)}$$

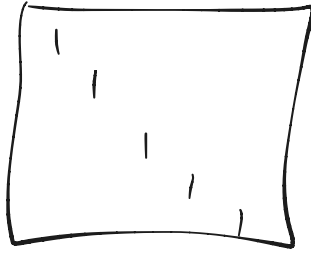
Log Rank Conjecture

$$\log_2(\text{Rank}(M_f)) \leq CC(f) \leq \log_2(\text{Rank}(M_f))^{O(1)}$$

Best known: $CC(f) = O(\sqrt{\text{rank}(M_f)})$ Lovett

$$CC(f) = \tilde{O}(\log(\text{rank}(M_f))^2) \quad \text{Jöös, P. Watso}$$

Observation $CC(EQ_n) = \Omega(n)$

$$M_{EQ_n} :$$


Pf 1 By logrank conj, M_{EQ_n} is identity matrix which has full rank $= 2^n$

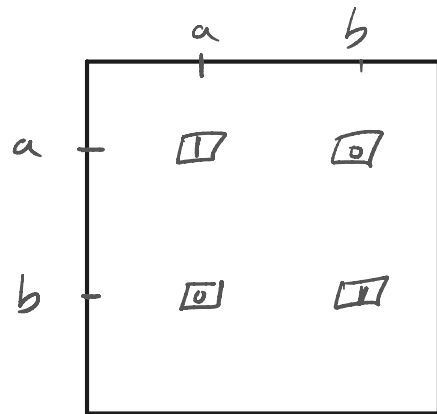
$$\therefore CC(EQ_n) = \Omega(n)$$

Observation $CC(EQ_n) = \Omega(n)$

Pf 2 "Fooling set" argument.

Claim: For any 2 distinct diagonal entries (a, a) , (b, b)

any protocol for EQ_n has (a, a) and (b, b) in different leaves.



Pf If not then since each leaf associated with a rectangle if (a, a) , (b, b) both in some leaf rectangle R_x then (a, b) , (b, a) also in R_x . But then R_x can't be monochromatic