Negative weights make adversaries stronger

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Quantum query complexity

- Popular model for study

- Seems to capture power of quantum computing:
  - Grover's search algorithm,
  - Period finding of Shor's algorithm,
  - Quantum walks: element distinctness, triangle finding, matrix multiplication

- And we can also prove lower bounds!
  - Polynomial method, Quantum Adversary method

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- All these methods shown equivalent by Špalek and Szegedy, 2006.

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- We essentially use that a successful algorithm *computes* a function, not just that it can *distinguish* inputs with different function values.

- Our method does not face the limitations of previous adversary methods.

Quantum queries

- In classical query complexity, want to compute $f(x)$ and can make queries of the form $x_i =?$ Complexity is number of queries on worst case input.

- Quantum query—turn query operator into unitary transformation on Hilbert Space $H_I \otimes H_Q \otimes H_W$

$$O|x\rangle|i\rangle|z\rangle \rightarrow (-1)^{x_i}|x\rangle|i\rangle|z\rangle.$$

- Can make queries in superposition.

Query algorithm

- On input x, algorithm proceeds by alternating queries and arbitrary unitary transformations independent of x

$$|\phi_x^t\rangle = U_t O U_{t-1} \ldots U_1 O U_0 |x\rangle|0\rangle|0\rangle.$$

- Output determined by complete set of orthogonal projectors $\{\Pi_0, \Pi_1\}$. A T-query algorithm outputs b on input x with probability $\|\Pi_b|\phi_x^T\rangle\|^2$.

- $Q_2(f)$ is number T of queries needed by best algorithm which outputs $f(x)$ on input x with probability at least $2/3$, for all x.

Matrix notation

- We will use matrix formulation of adversary method [BSS03]
- Spectral norm $\|A\| = \sqrt{\lambda_1(AA^*)}$.
- Hadamard (entrywise) product $(A \circ B)[i,j] = A[i,j] \cdot B[i,j]$.

Adversary method

Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function, and Γ a symmetric 2^n-by-2^n matrix where $\Gamma[x,y] = 0$ if $f(x) = f(y)$. Then

$$\mathrm{ADV}(f) = \max_{\substack{\Gamma \geq 0 \\ \Gamma \neq 0}} \frac{\|\Gamma\|}{\max_i \|\Gamma \circ D_i\|}.$$

D_i is a zero-one matrix where $D_i[x,y] = 1$ if $x_i \neq y_i$ and $D_i[x,y] = 0$ otherwise.

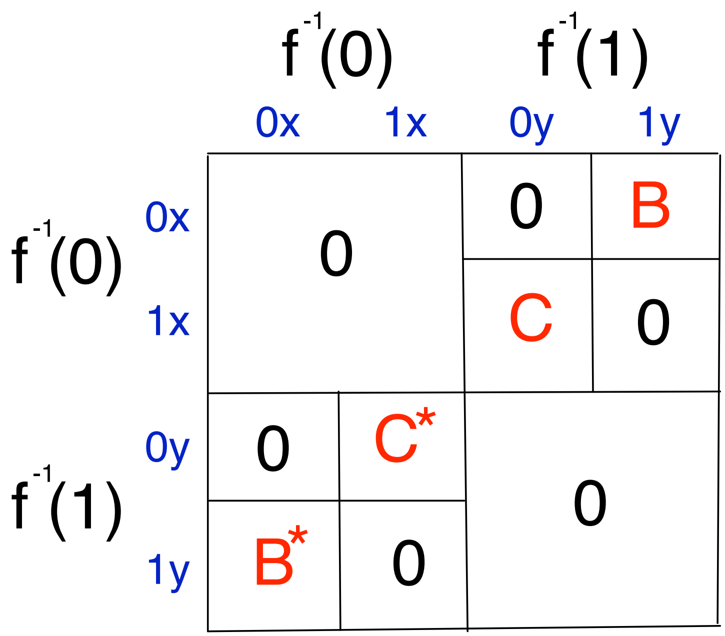
Theorem [BSS03]: $Q_2(f) = \Omega(\mathrm{ADV}(f))$.

The Γ matrix

	$f^{-1}(0)$	$f^{-1}(1)$
$f^{-1}(0)$	0	A
$f^{-1}(1)$	A^*	0

Notice that the spectral norm of Γ equals that of A.

The $\Gamma \circ D_1$ matrix



The spectral norm of $\Gamma \circ D_1$ equals $\max\{\|B\|, \|C\|\}$.

Example: OR function

We define the matrix:

	1000	0100	0010	0001
0000	1	1	1	1

The spectral norm of this matrix is $\sqrt{4}$, and the spectral norm of each $\Gamma \circ D_i$ is one.

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Generalizing this construction we find $Q_2(\mathrm{OR}_n) = \Omega(\sqrt{n})$.

New adversary method

We remove the restriction to nonnegative matrices:

$$\mathrm{ADV}^{\pm}(f) = \max_{\Gamma \neq 0} \frac{\|\Gamma\|}{\max_i \|\Gamma \circ D_i\|}.$$

Theorem: $Q_2(f) = \Omega(\mathrm{ADV}^{\pm}(f))$.

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Theorem: $Q_2(f) = \Omega(\mathrm{ADV}^{\pm}(f))$.

As we maximize over a larger set, $\mathrm{ADV}^{\pm}(f) \geq \mathrm{ADV}(f)$. It turns out that negative entries can help in giving larger lower bounds!

Separating the old and new

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- Given a list of n elements in $\{1, 2, \ldots, n\}$, are they all distinct? $\mathrm{ADV}(f) \leq \sqrt{2n}$, and right answer is $\Theta(n^{2/3})$ [AS04, Amb04].

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- We have example where $\mathrm{ADV}^{\pm}(f) = \Omega((C_0(f)C_1(f))^{0.549})$.

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Recall that running the algorithm on input x for t queries:

$$|\phi_x^t\rangle = U_t O U_{t-1} \dots U_1 O U_0 |x\rangle|0\rangle|0\rangle.$$

Write this as $|\phi_x^t\rangle = |x\rangle|\psi_x^t\rangle$.

Let Γ be an adversary matrix and δ a principal eigenvector.

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Let Γ be an adversary matrix and δ a principal eigenvector. The principal eigenvector *tells us* how to build a hard input— we feed algorithm the superposition $\sum_x \delta_x |x\rangle|0\rangle|0\rangle$. State of algorithm after t queries is $\sum_x \delta_x |x\rangle|\psi_x^t\rangle$. Let $\rho^{(t)}[x,y] = \delta_x^* \delta_y \langle \psi_x^t | \psi_y^t \rangle$ be the reduced density matrix of this state.

Watch the density matrix. . .

Define a progress function based on $\rho^{(t)}$ as

$$W^{(t)} = \langle \Gamma, \rho^{(t)} \rangle = \sum_{x,y} \Gamma[x,y] \delta_x^* \delta_y \langle \psi_x^t | \psi_y^t \rangle.$$

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Show three things:

- $|W^{(0)}| = \|\Gamma\|$

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- $|W^{(0)}| = \|\Gamma\|$
- $|W^{(T)}| \leq 2\sqrt{\epsilon(1-\epsilon)} \|\Gamma\|$
- $|W^{(t)} - W^{(t+1)}| \leq 2 \max_i \|\Gamma \circ D_i\|$

Step Two: Old adversary

- Want to upper bound $\langle \Gamma, \rho^{(T)} \rangle \leq 2\sqrt{\epsilon(1-\epsilon)}\|\Gamma\|$.

- Distinguishing principle: Successful algorithm can distinguish 0-inputs from 1-inputs with error probability ϵ means

$$\langle \psi_x^T | \psi_y^T \rangle \leq 2\sqrt{\epsilon(1-\epsilon)}$$

- Thus as Γ nonnegative

$$\begin{aligned} \sum_{x,y} \Gamma[x,y]\delta_x^* \delta_y \langle \psi_x^T | \psi_y^T \rangle &\leq 2\sqrt{\epsilon(1-\epsilon)} \sum_{x,y} \Gamma[x,y]\delta_x^* \delta_y \\ &= 2\sqrt{\epsilon(1-\epsilon)}\|\Gamma\| \end{aligned}$$

User's Manual

- Automorphism principle: If π is automorphism of the function then wlog, $\Gamma[x,y] = \Gamma[\pi(x),\pi(y)]$ in optimal adversary matrix.

- Composition principle: Let $f : \{0,1\}^n \rightarrow \{0,1\}$. Write $f^1 = f$ and $f^d : \{0,1\}^{n^d} \rightarrow \{0,1\}$ be

$$f^d(x) = f(f^{d-1}(x^{(1)}), f^{d-1}(x^{(2)}), \ldots, f^{d-1}(x^{(n)})),$$

where $x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$. Then $\mathrm{ADV}^{\pm}(f^d) \geq \mathrm{ADV}^{\pm}(f)^d$.

Another example: Ambainis function

- Originally used by Ambainis to separate quantum query complexity from polynomial degree.
- Automorphism group isomorphic to $\mathbb{Z}_8$, generated by $(4321) \times (0,0,0,1)$.
- The zeros: 0000

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	0010	0101	1011	0110	1101	1010	0100	1001
0000								
0001								
0011								
0111								
1111								
1110								
1100								
1000								

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	0010	0101	1011	0110	1101	1010	0100	1001
0000	a							
0001								
0011								
0111								
1111								
1110								
1100								
1000								

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	0010	0101	1011	0110	1101	1010	0100	1001
0000	a							
0001		a						
0011			a					
0111				a				
1111					a			
1110						a		
1100							a	
1000								a

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	0010	0101	1011	0110	1101	1010	0100	1001
0000	a	c						
0001		a						
0011			a					
0111				a				
1111					a			
1110						a		
1100							a	
1000								a

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1111					a	c		
1110						a	c	
1100							a	c
1000	c							a

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	0010	0101	1011	0110	1101	1010	0100	1001
0000	a	c	d					
0001		a	c	d				
0011			a	c	d			
0111				a	c	d		
1111					a	c	d	
1110						a	c	d
1100	d						a	c
1000	c	d						a

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0000	a	c	d	b				
0001		a	c	d	b			
0011			a	c	d	b		
0111				a	c	d	b	
1111					a	c	d	b
1110	b					a	c	d
1100	d	b					a	c
1000	c	d	b					a

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0000	a	c	d	b	d			
0001		a	c	d	b	d		
0011			a	c	d	b	d	
0111				a	c	d	b	d
1111	d				a	c	d	b
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1100	d	b	d				a	c
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1000	c	d	b	d	c			a

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0111	c	a		a	c	d	b	d
1111	d	c	a		a	c	d	b
1110	b	d	c	a		a	c	d
1100	d	b	d	c	a		a	c
1000	c	d	b	d	c	a		a

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0000	a	c	d	b	d	c	a	b
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0011	a	b	a	c	d	b	d	c
0111	c	a	b	a	c	d	b	d
1111	d	c	a	b	a	c	d	b
1110	b	d	c	a	b	a	c	d
1100	d	b	d	c	a	b	a	c
1000	c	d	b	d	c	a	b	a

The $\Gamma \circ D_1$ matrix

	1001	1010	1011	1101
0011	c	b	a	d
0000	b	c	d	d
0001	a	d	c	b
0111	d	d	b	c

Ambainis function continued

We try to maximize $\|\Gamma\| = 2(a+b+c+d)$ while keeping spectral norm of $\Gamma \circ D_i$ at most 1.

	a	b	c	d	$\|\Gamma\|$
ADV	0.75	0.50	0	0	2.5
$\mathrm{ADV}^{\pm}$	0.5788	0.7065	0.1834	-0.2120	2.5136

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The Ambainis function has polynomial degree 2. By iterating this function, we obtain largest known separation between polynomial degree and quantum query complexity, m vs $m^{1.327}$.

Open Questions

- Element distinctness: Best bound provable by old method is $\sqrt{2n}$, but right answer is $n^{2/3}$, provable by polynomial method. Can new adversary method prove optimal bound?

- Triangle finding: Best bound provable by old method is n, and best known algorithm gives $n^{1.3}$. Can new adversary bound give a superlinear lower bound?

- $\mathrm{ADV}^{\pm}(f)^2$ is a lower bound on the formula size of f. Conjecture: The bounded-error quantum query complexity of f squared is, in general, a lower bound on the formula size of f.