Matrix Methods for Formula Size Lower Bounds

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Circuit Complexity

- A million dollar question: Show an explicit function (in NP) which requires superpolynomial size circuits!
- For functions in NP the best circuit lower bound we know is $5n - o(n)$ [LR01, IM02]
- The smallest complexity class we know to contain a function requiring superpolynomial size circuits is MAEXP! [BFT98]

Formula Size

- Weakening of the circuit model—a formula is a binary tree with internal nodes labelled by AND, OR and leaves labelled by literals. The size of a formula is its number of leaves.

- PARITY has formula size $\theta(n^2)$ [Khr71].

- Showing superpolynomial formula size lower bounds for a function in NP would imply NP $\neq$ NC^1.

- The best lower bound for a function in NP is $n^{3-o(1)}$ [Hås98].

A New Technique

- We devise a new lower bound technique based on matrix rank.

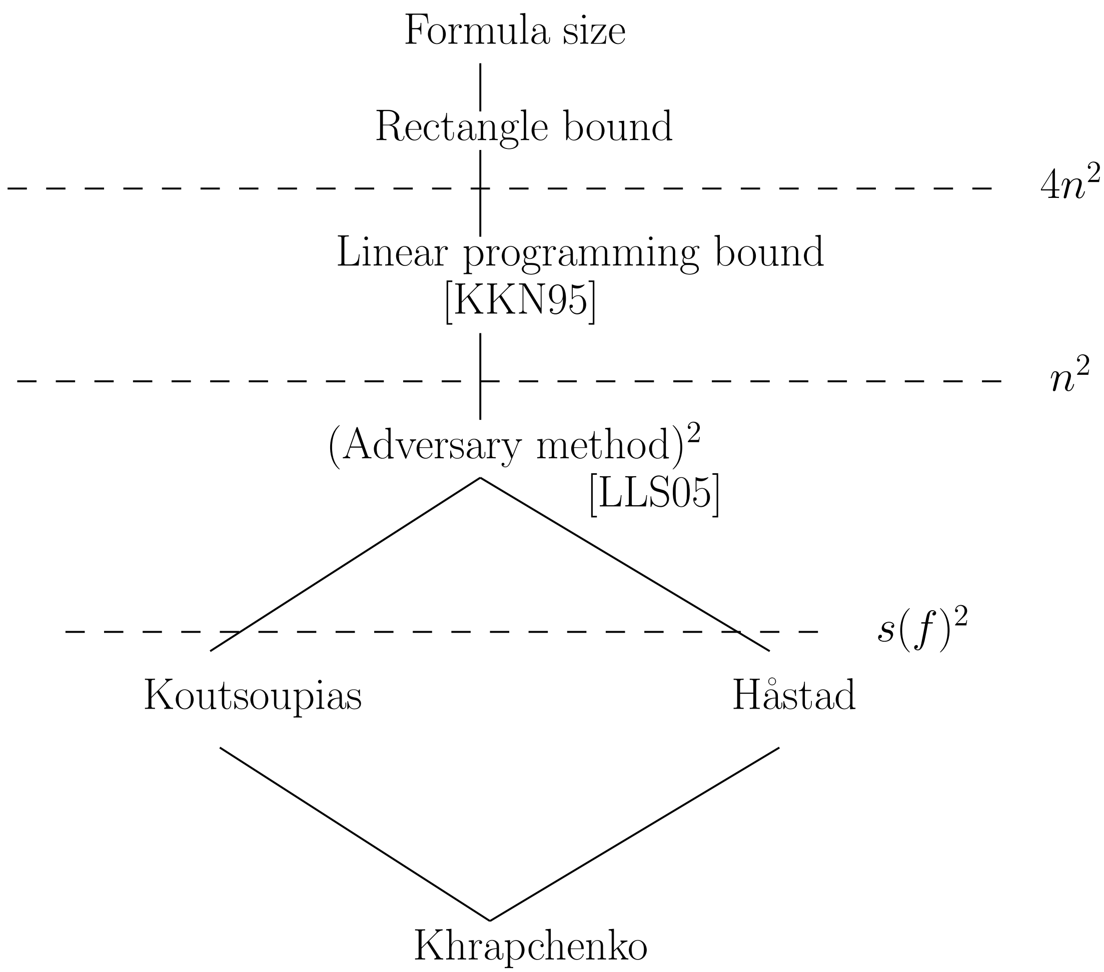
- We exactly determine the formula size of PARITY: if $n = 2^\ell + k$ then

$$L(\text{PARITY}) = 2^\ell(2^\ell + 3k) = n^2 + k2^\ell - k^2.$$

- The formula size of many other basic functions remains unresolved:

$$\frac{n^2}{4} \leq L(\text{MAJORITY}) \leq n^{4.57}$$

A Hierarchy of Techniques



Karchmer–Wigderson Game [KW88]

- Elegant characterization of formula size in terms of a communication game.

- For a Boolean function f, let $X = f^{-1}(0), Y = f^{-1}(1)$ and

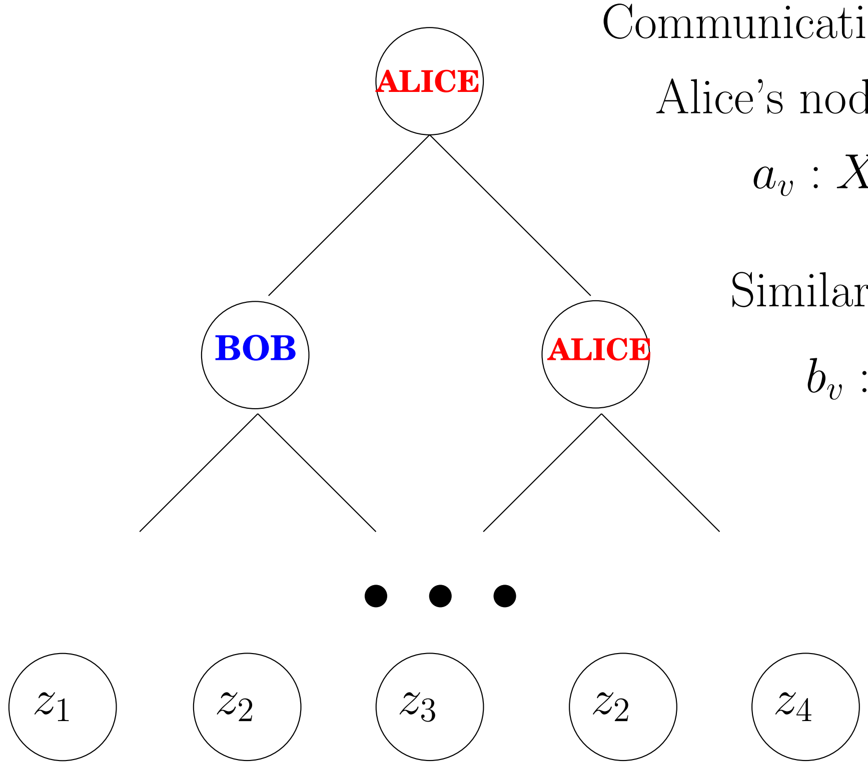
$$R_f = \{(x, y, i) : x \in X,\ y \in Y,\ x_i \neq y_i\}$$

- The game is then the following: Alice is given $x \in X$, Bob is given $y \in Y$ and they wish to find i such that $(x, y, i) \in R_f$.

- Karchmer–Wigderson Thm: The number of leaves in a best communication protocol for R_f equals the formula size of f.

Communication complexity of relations

$R \subseteq X \times Y \times Z$



Communication protocol is a binary tree:

Alice's nodes labelled by a function:

$a_v : X \to \{0, 1\}$

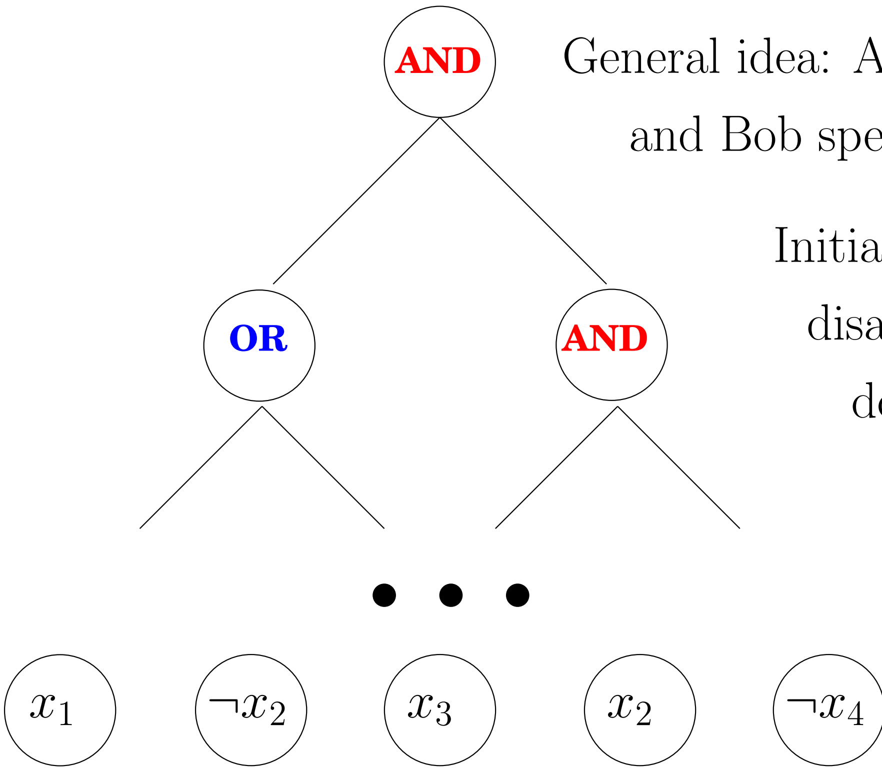
Similarly, Bob's nodes labelled

$b_v : Y \to \{0, 1\}$

Leaves labelled by elements $z \in Z$.

Denote by $C^P(R)$ the number of leaves in a best protocol for R.

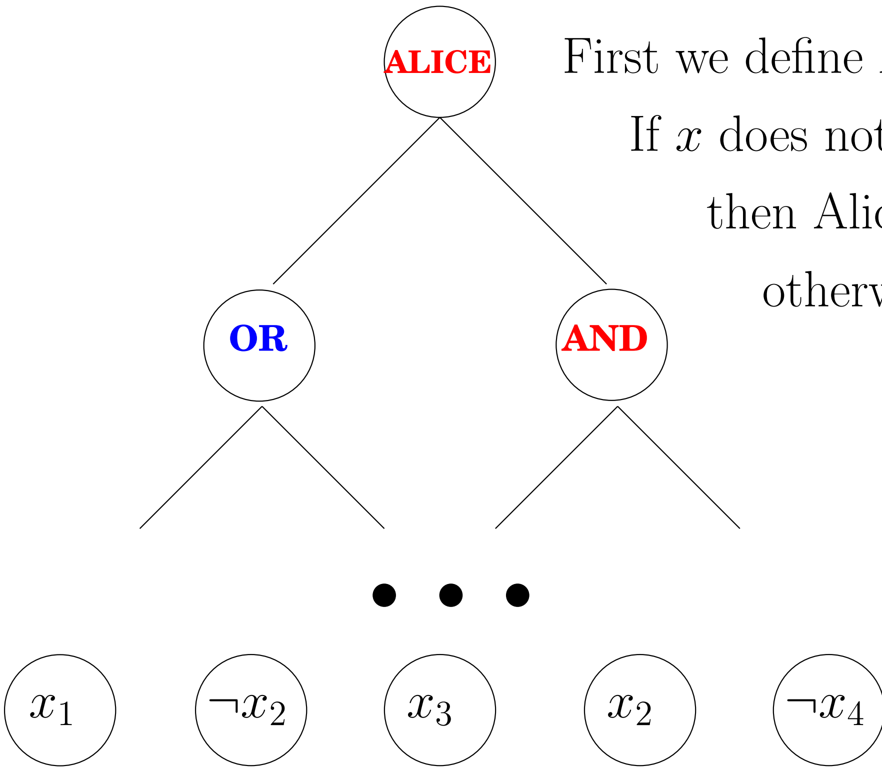
Proof by picture: $C^P(R_f) \leq \mathsf{L}(f)$.



General idea: Alice speaks at AND nodes and Bob speaks at OR nodes.

Initially, $f(x) \neq f(y)$ and we maintain this disagreement on subformulas as we move down the tree.

Proof by picture: $C^P(R_f) \leq \mathsf{L}(f)$.



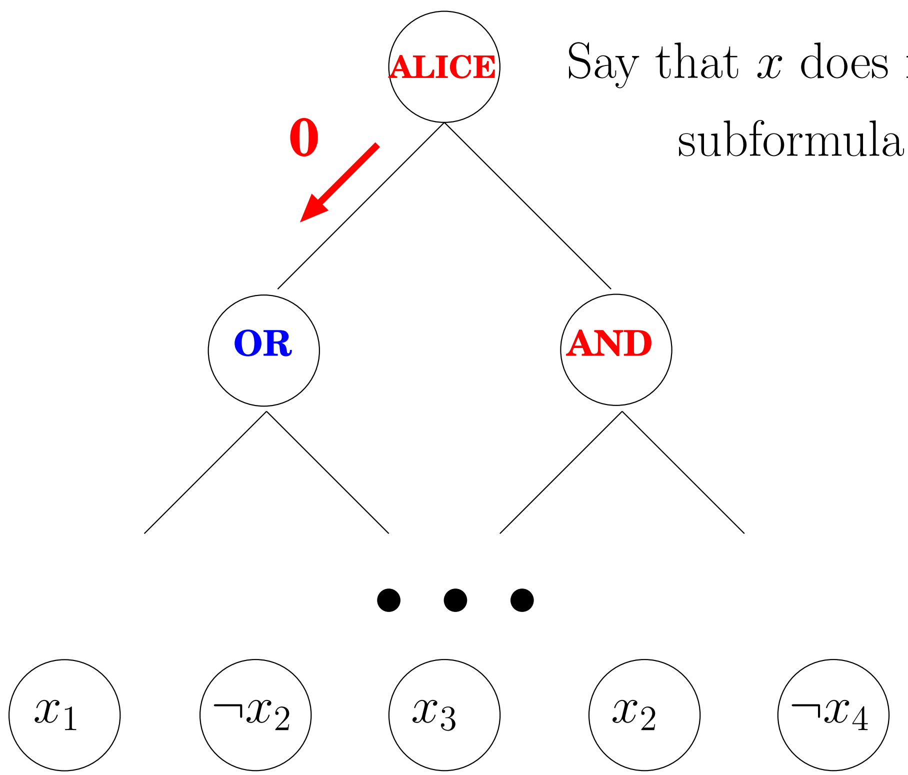
First we define Alice's action at the top node:

If x does not satisfy the left subformula,

then Alice sends the bit 0;

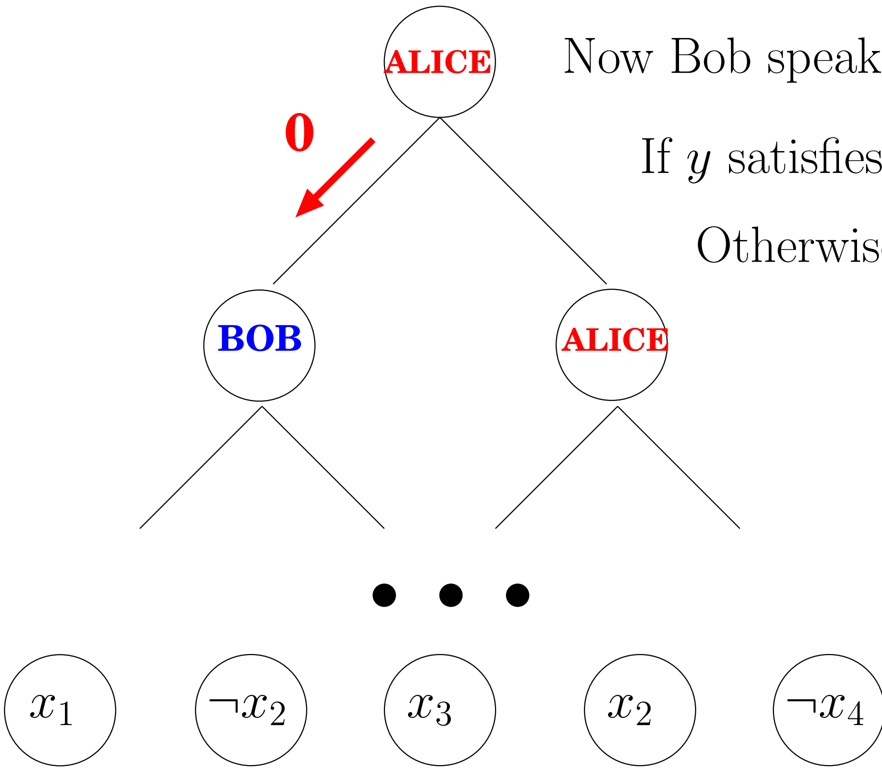
otherwise she sends the bit 1.

Proof by picture: $C^P(R_f) \leq \mathsf{L}(f)$.



Say that x does not satisfy the left subformula.

Proof by picture: $C^P(R_f) \leq \mathsf{L}(f)$.

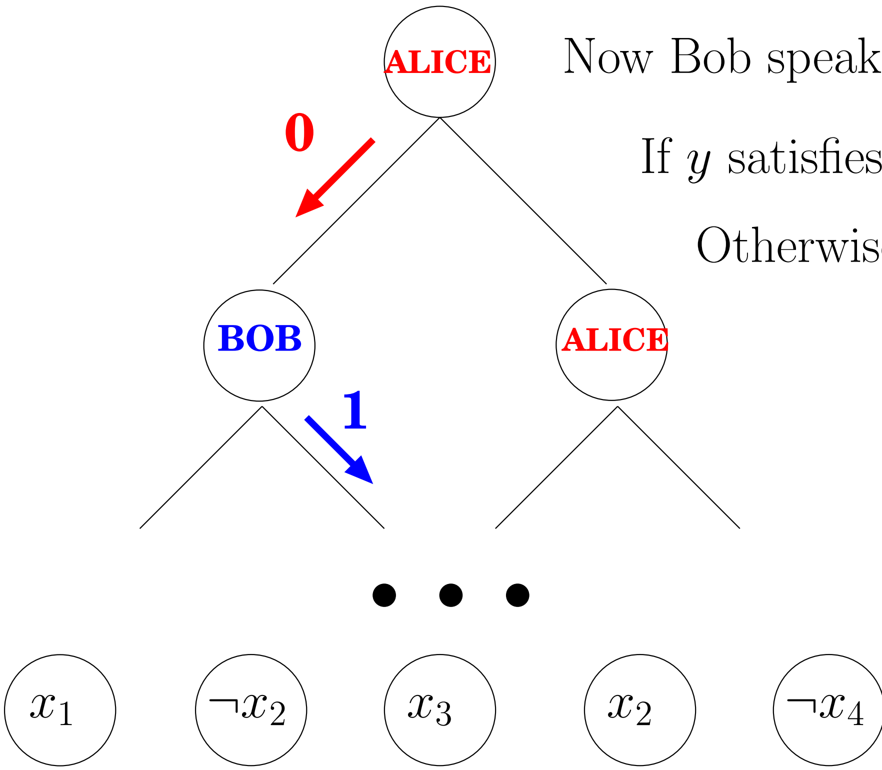


Now Bob speaks at the OR gate:

If y satisfies the left subformula, Bob says 0.

Otherwise, he says 1.

Proof by picture: $C^P(R_f) \leq \mathsf{L}(f)$.



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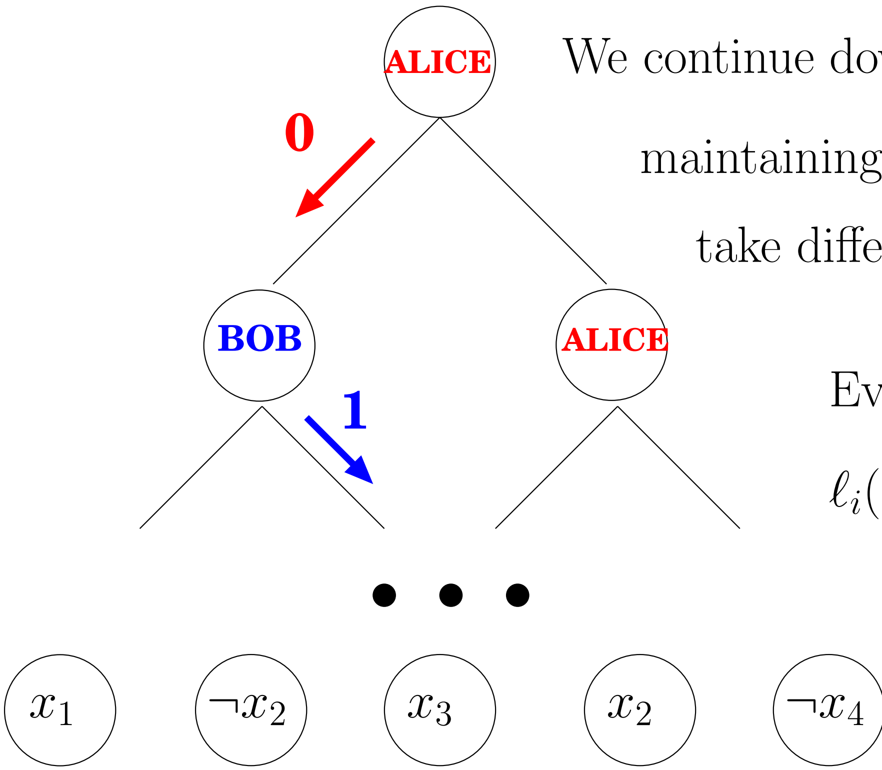
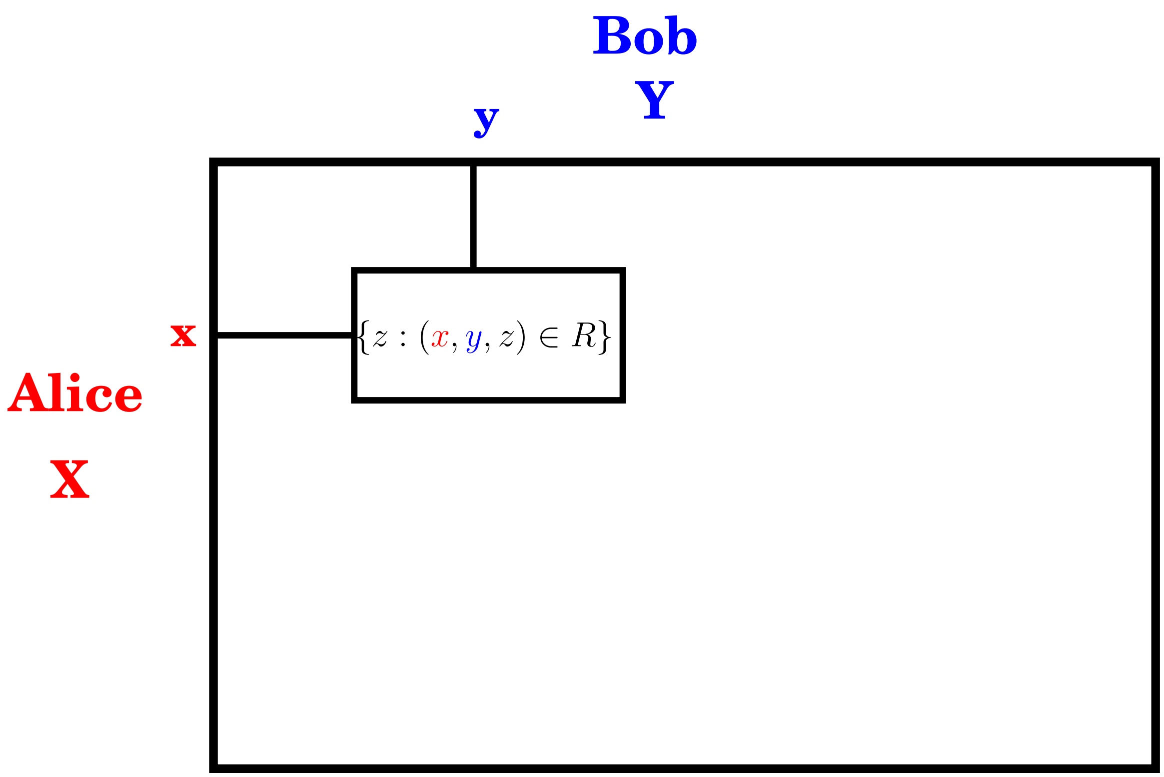


We continue down the tree in a similar fashion, maintaining the property that x and y take different values on subformulas.

Eventually, we reach a literal ℓ_i such that $\ell_i(x) \neq \ell_i(y)$ and so x and y differ on bit i.

Communication Complexity and the Rectangle Bound

$$R \subseteq X \times Y \times Z$$



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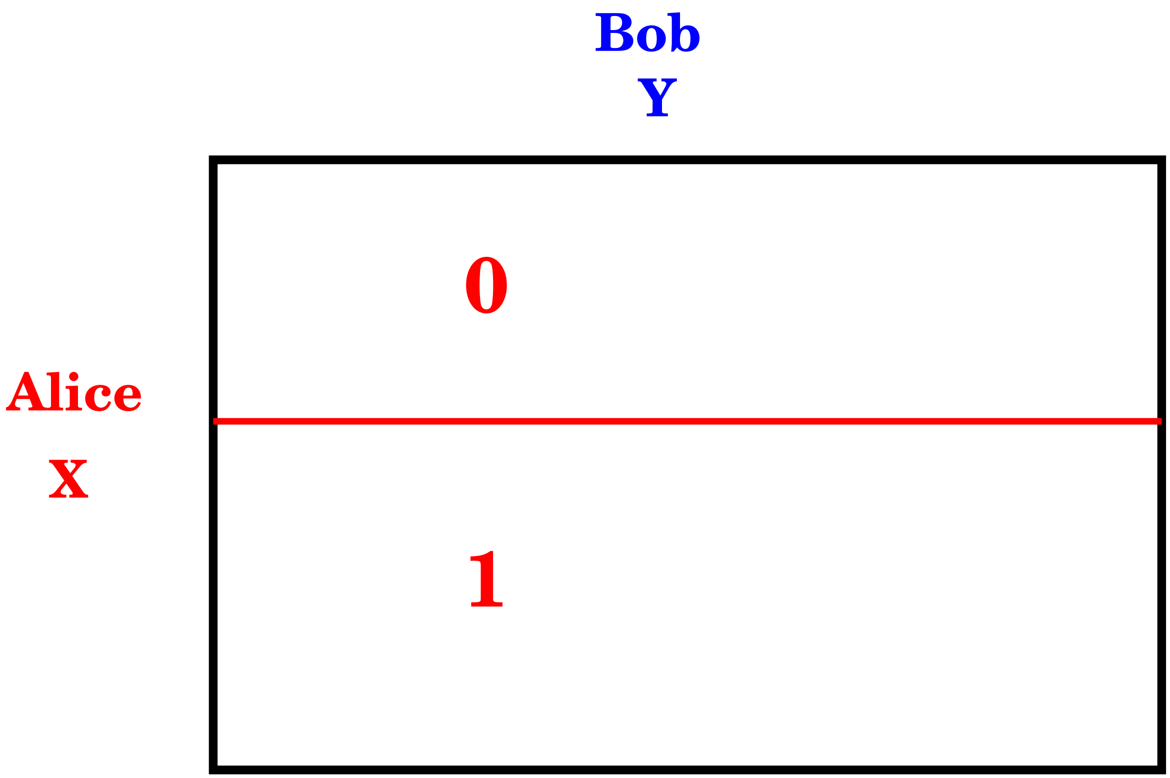

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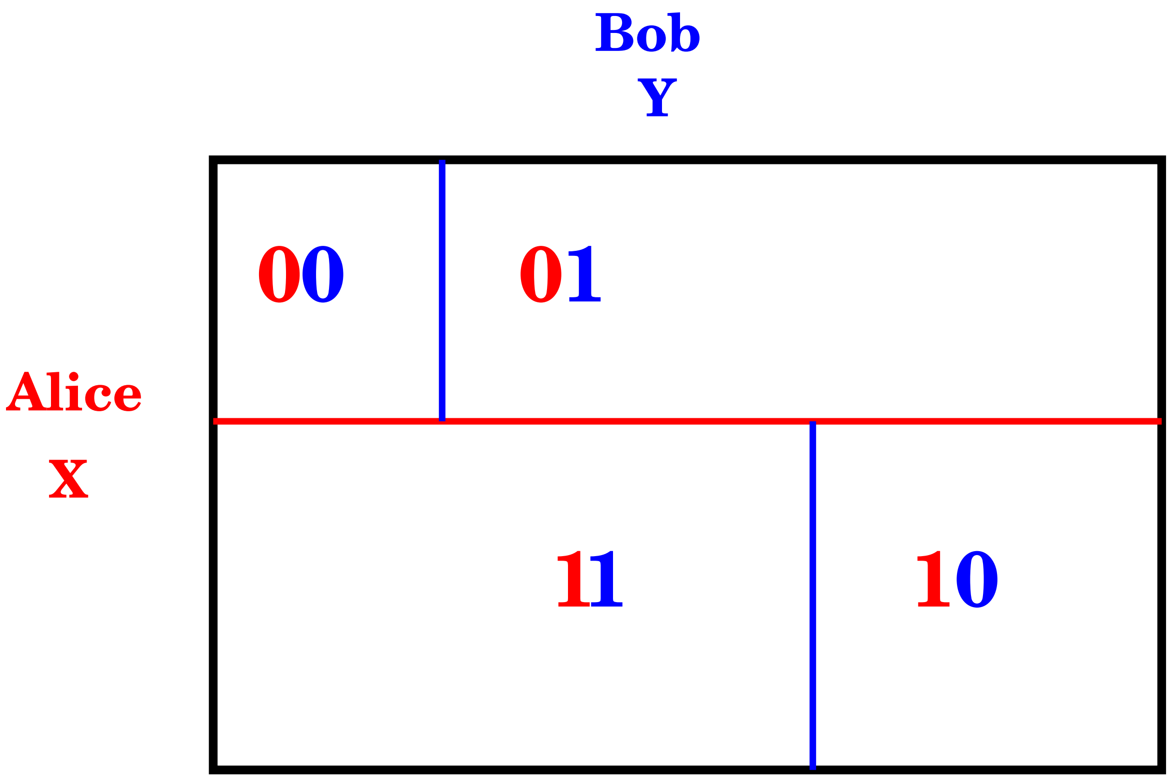

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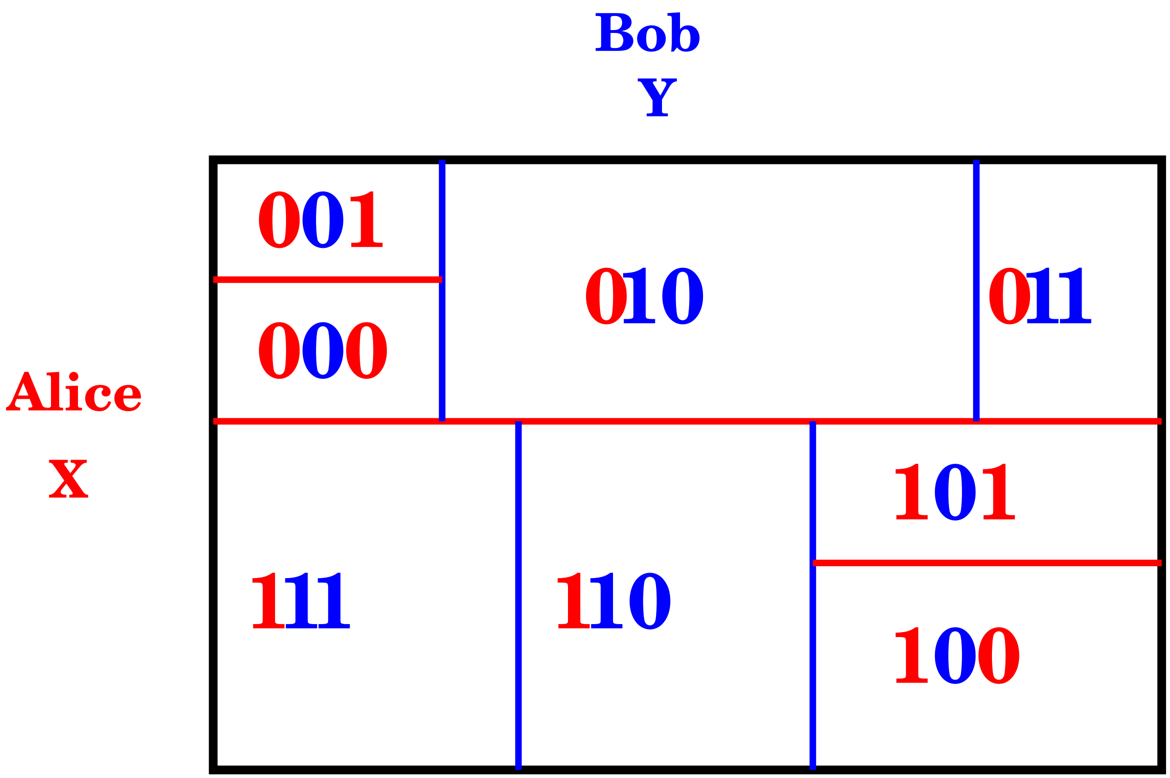

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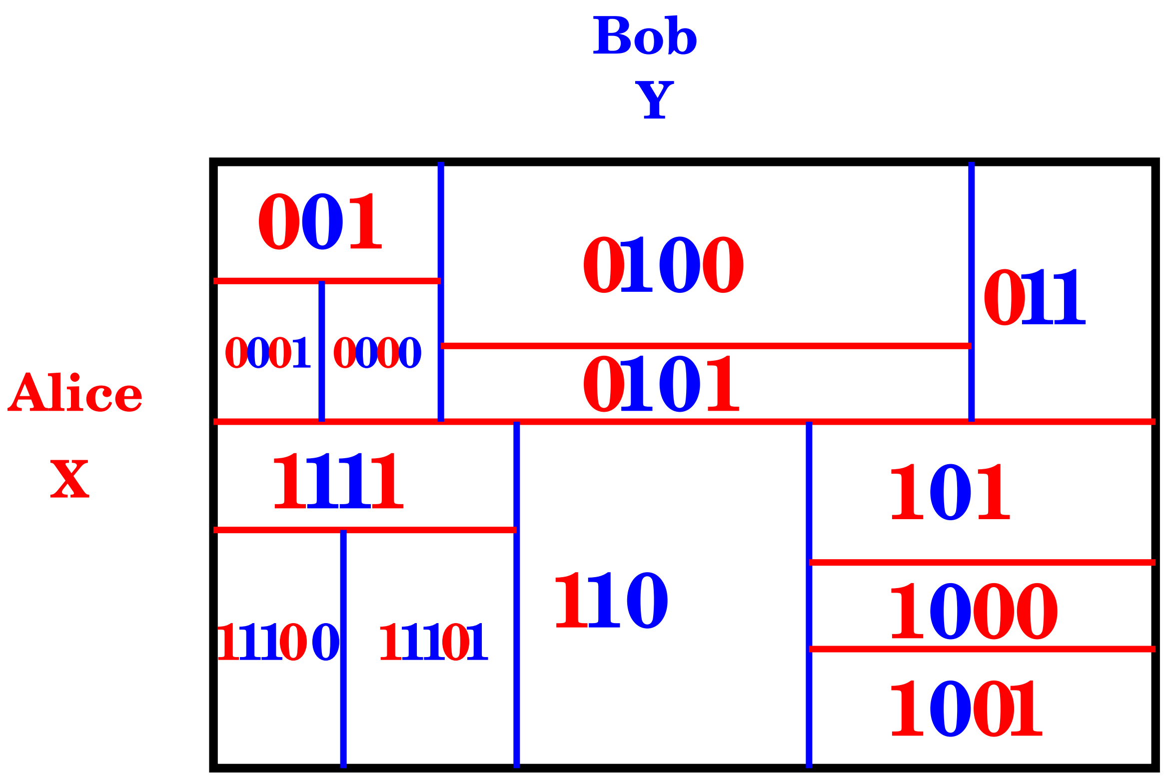


A rectangle S is monochromatic if there exists z such that $(x, y, z) \in S$ for all $(x, y) \in S$.

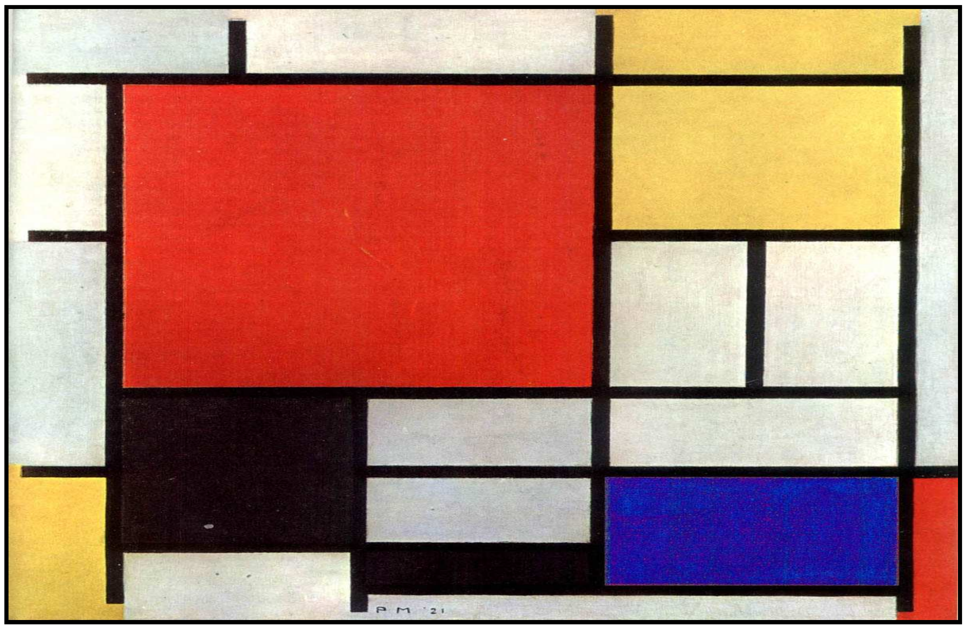
A successful protocol partitions $X \times Y$ into monochromatic rectangles.

Communication Complexity and the Rectangle Bound

$$R \subseteq X \times Y \times Z$$

Bob
Y

Alice
X



Rectangle Bound

- We denote by $C^D(R)$ the size of a smallest partition of $X \times Y$ into monochromatic (with respect to R) rectangles. By the argument above, $C^D(R) \leq C^P(R)$.

- The rectangle bound is a purely combinatorial quantity.

- We can still hope to prove larger lower bounds by focusing on the rectangle bound:

$$C^D(R) \leq C^P(R) \leq 2^{(\log C^D(R))^2}$$

- Major drawback—it is NP hard to compute.

Rectangles and Rank

- Rank is one of the most successful ways to prove lower bounds on communication complexity of functions

- Let $M[x, y] = f(x, y)$. A monochromatic 1-rectangle has rank one, thus $\text{rk}(M) \leq C^D(f)$.

- It has been difficult to adapt the rank technique to communication complexity of relations.

Rank for relations

- The key idea is a selection function $S : X \times Y \to Z$.
- A selection function turns a relation into a function, by selecting one output.
- Let $R|_S = \{(x, y, z) : S(x, y) = z\}$. Then
$$C^P(R) = \min_S C^P(R|_S).$$

Rank for relations

- With the help of selection functions, we can now apply the rank method as before.

- Let S_z be a matrix where $S_z[x, y] = 1$ if $S(x, y) = z$ and 0 otherwise.

$$\min_S \sum_{z \in Z} \mathrm{rk}(S_z) \leq C^D(R)$$

Approximating Rank

- In general this bound seems difficult to use because of the minimization over all selection functions

- We get around this by the following lower bound on rank:

$$\left\lceil \frac{\|M\|_{\mathrm{tr}}^2}{\|M\|_F^2} \right\rceil \leq \mathrm{rk}(M)$$

  where

  - $\|M\|_{\mathrm{tr}} = \sum_i \lambda_i(M)$
  - $\|M\|_F^2 = \sum_i \lambda_i^2(M)$

Application to Parity

- Selection function: $S : 2^{n-1} \times 2^{n-1} \to [n]$.

- For every $i \in [n]$, there are 2^{n-1} pairs where behavior of selection function is determined—the sensitive pairs.

- If selection function S only output i where forced to, then $\mathrm{rk}(S_i) = 2^{n-1}$. Thus S must output i in more places to bring down rank.

Application to Parity

- Because of sensitive pairs $\|S_i\|_{\mathrm{tr}} \geq 2^{n-1}$ for every i.

- Also, $\|S_i\|_F^2$ is simply number of ones in S_i.

- Putting these observations together:

$$\min_{s_i} \sum_i \left\lceil \frac{(2^{n-1})^2}{s_i} \right\rceil \leq L(\mathrm{PARITY})$$

  where $\sum_i s_i = (2^{n-1})^2$.

Application to Parity

We have

$$\min_{s_i} \sum_i \left\lceil \frac{(2^{n-1})^2}{s_i} \right\rceil \leq L(\text{PARITY})$$

where $\sum_i s_i = (2^{n-1})^2$.

- Ignoring the ceilings, Jensen's inequality says minimum attained when all s_i equal, $s_i = (2^{n-1})^2/n$. This is not possible when n is not a power of two.

- If $n = 2^\ell + k$, best thing to do, take each s_i a power of two, as evenly as possible:

$$L(\text{PARITY}) = 2^\ell(2^\ell + 3k) = n^2 + k2^\ell - k^2$$

Open problems

- Application to threshold functions?

$$\frac{n^2}{4} \leq L(\text{MAJORITY}) \leq n^{4.57}$$

- More subtle lower bound on rank? Use not just number of ones in each S_i but also their placement.