An Approximation Algorithm for Approximation Rank

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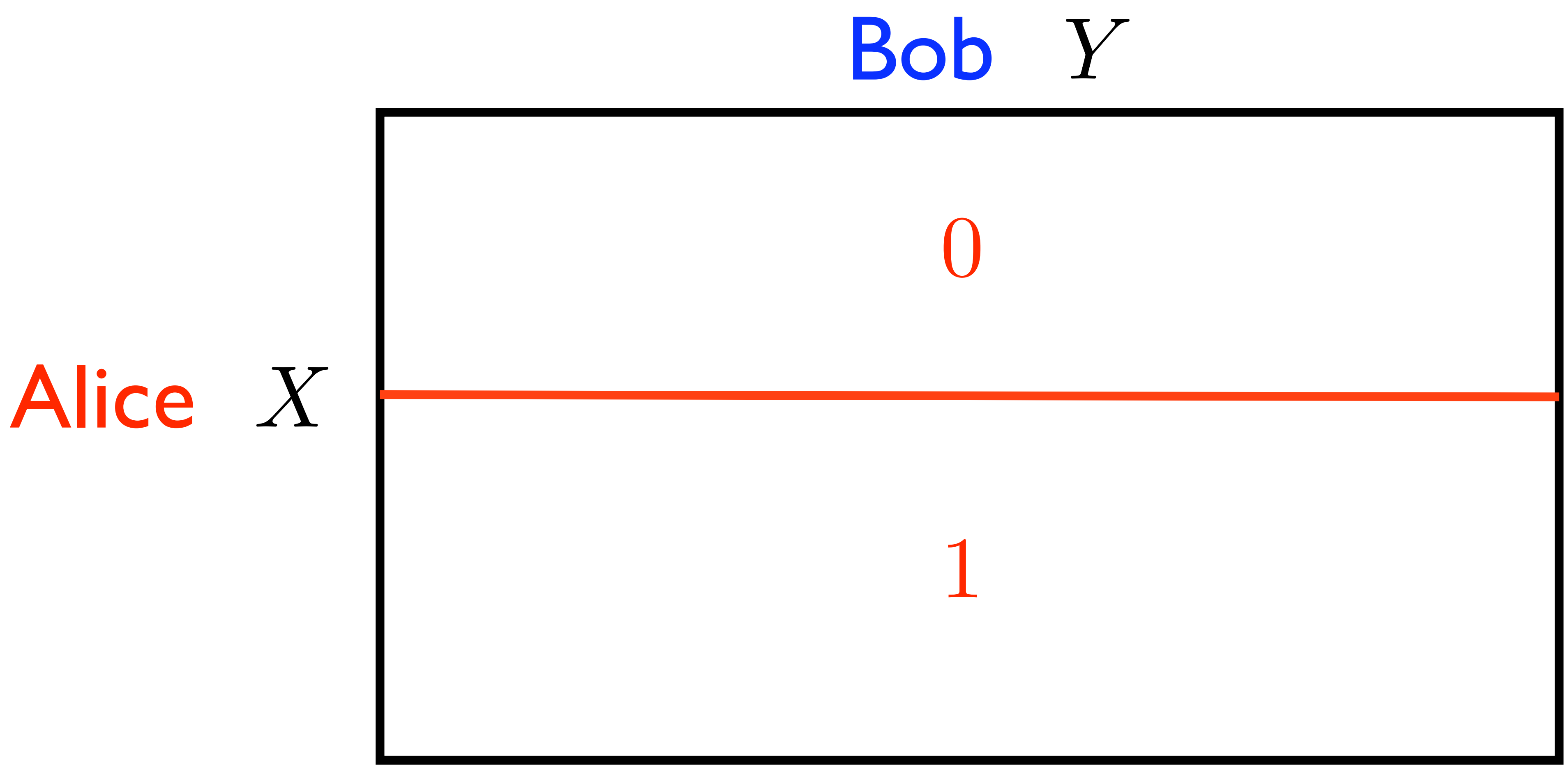
Conventions

Identify a communication function $f : X \times Y \to \{-1, +1\}$ with the associated X-by-Y matrix A(x,y)=f(x,y).

Denote by D(A) the deterministic communication complexity of the sign matrix A.

Denote by $R_\epsilon(A)$ the randomized complexity with error at most ϵ.

How a protocol partitions a matrix



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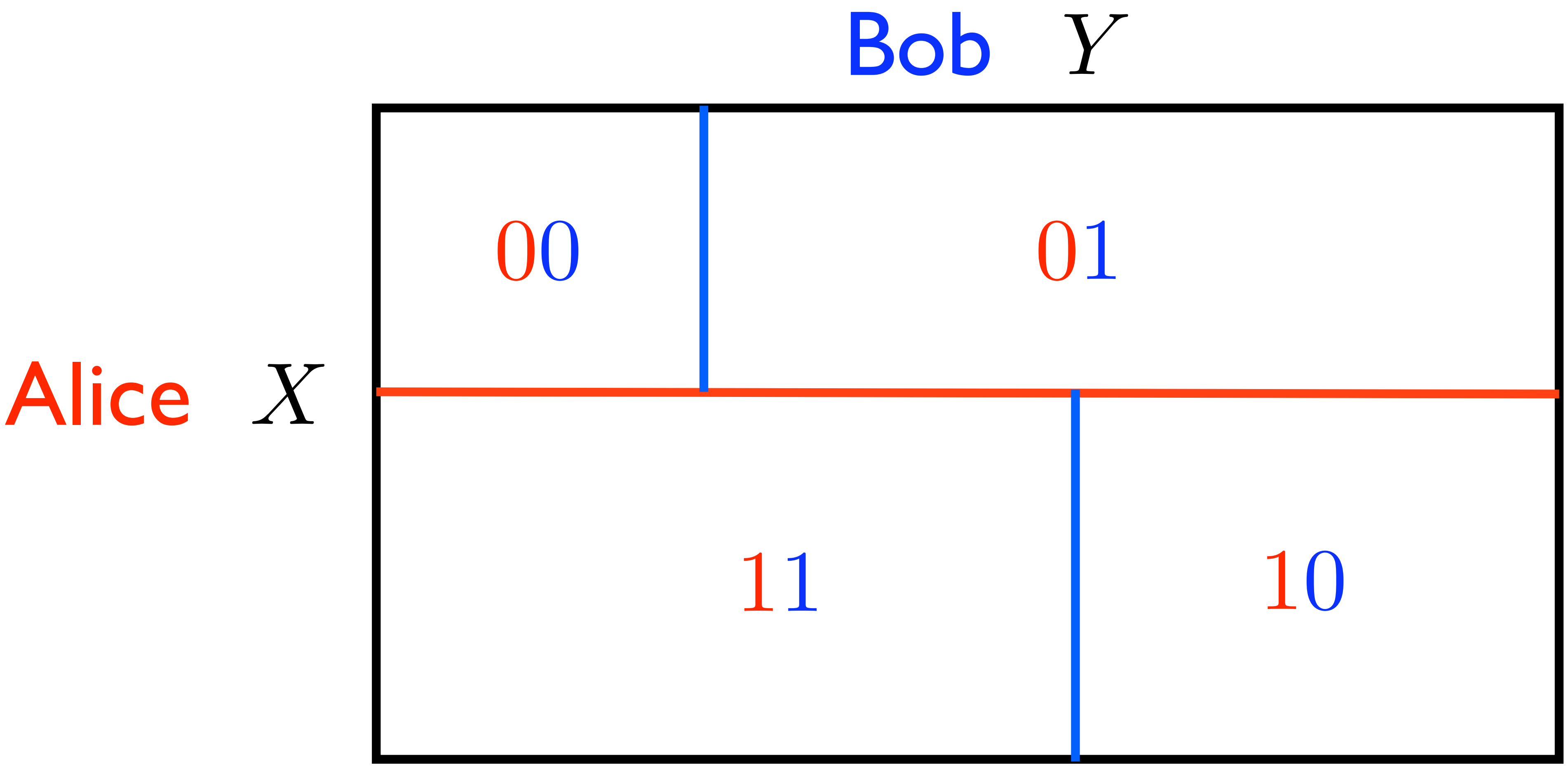

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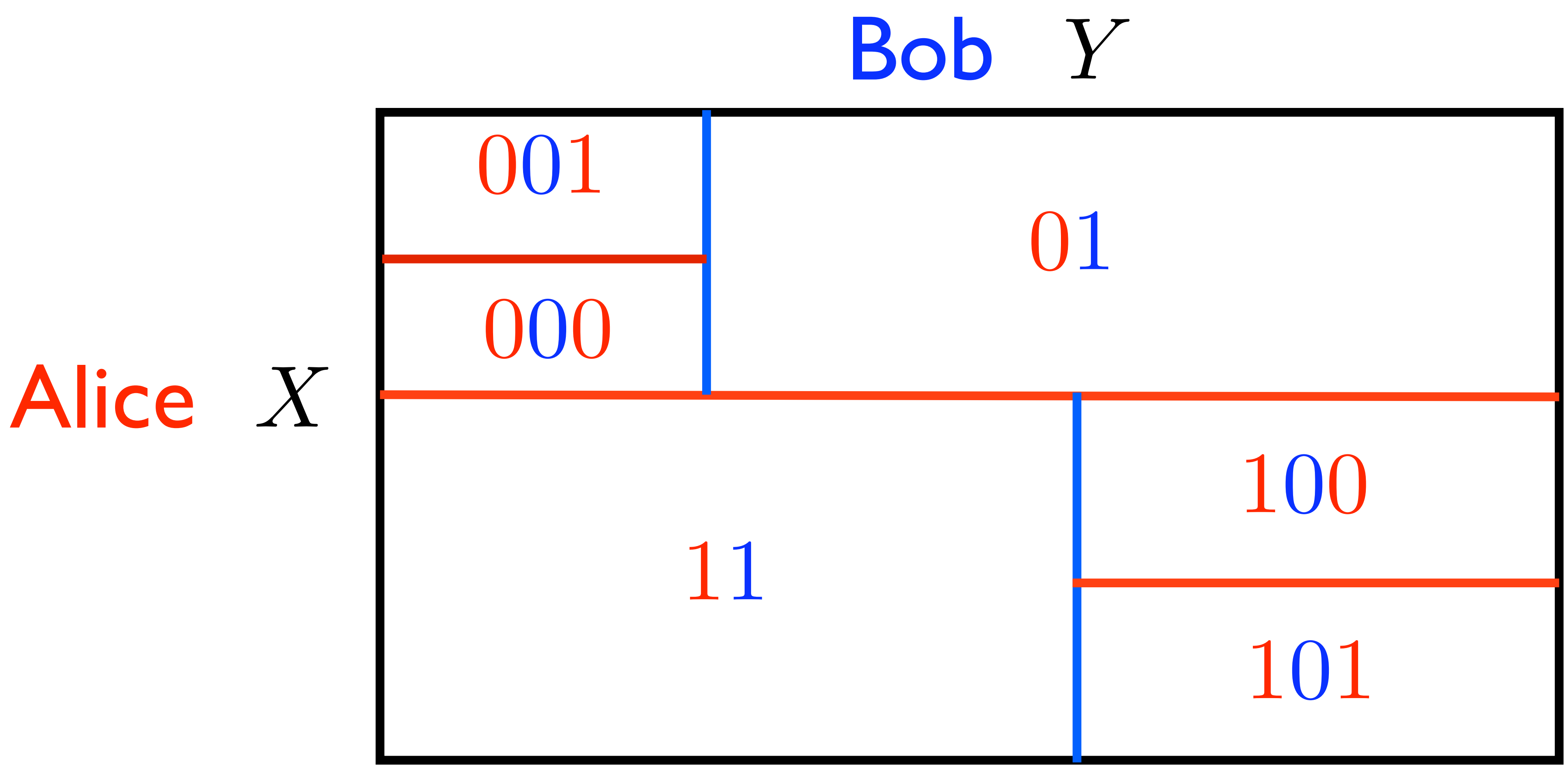

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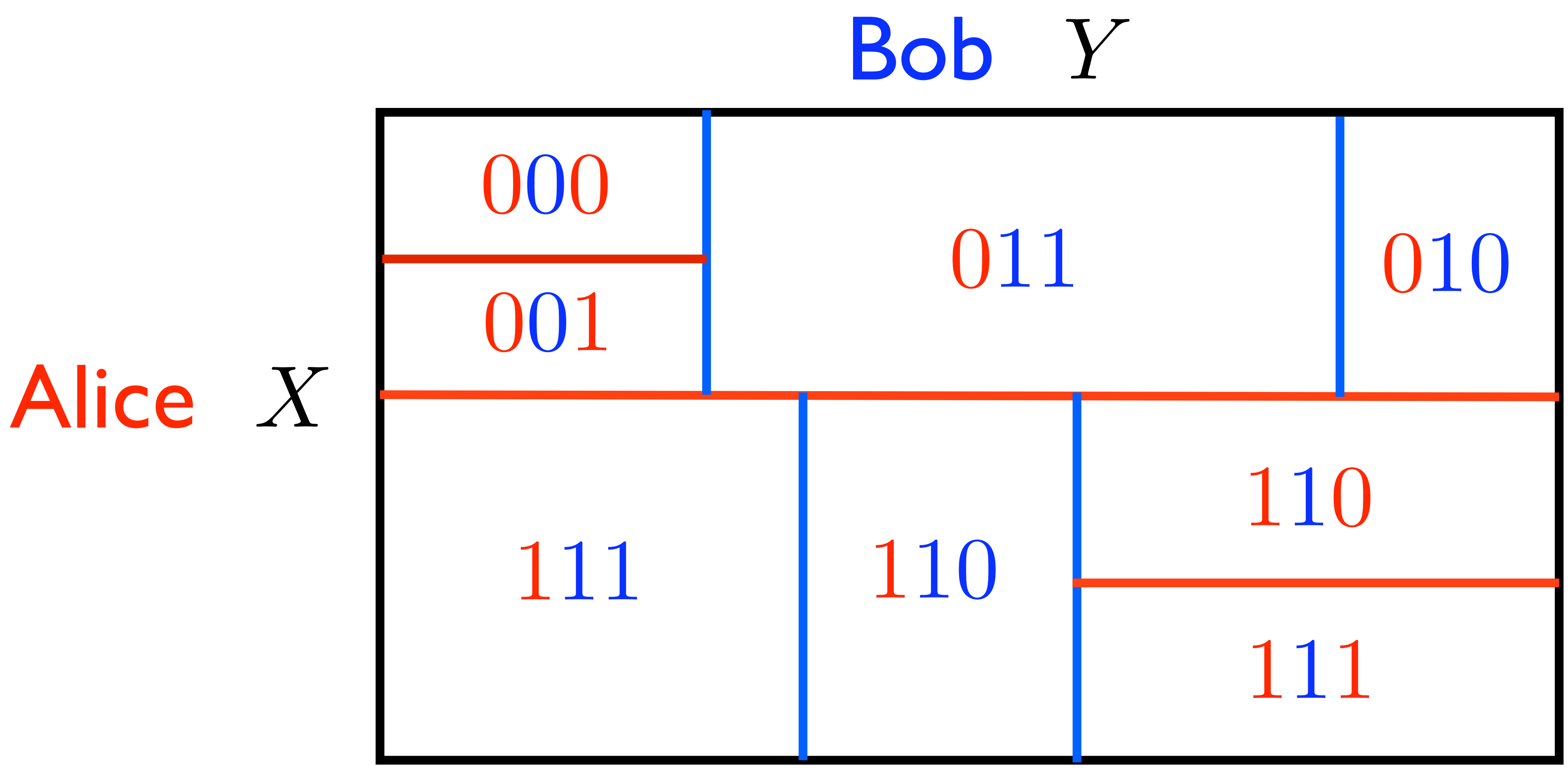

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One of the greatest open problems in communication complexity, the log rank conjecture of Lovasz and Saks, states that this bound is polynomially tight

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Given a target matrix A, approximation rank looks at the minimal rank matrix entrywise close to A:

$$
\begin{aligned}
\mathrm{rank}_\epsilon(A) \quad &= \quad \min_X \mathrm{rank}(X) \\
& \quad\quad |X(i,j) - A(i,j)| \leq \epsilon \text{ for all } i,j.
\end{aligned}
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While approximation rank gives a lower bound on quantum communication complexity, not known to work for model with entanglement.

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Moreover, this quantity is known to be a lower bound even in the quantum model with entanglement.

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For this, we need to introduce some matrix norms.

Matrix norms

- Define the i[th] singular value as

$$\sigma_i(A) = \sqrt{\lambda_i(AA^t)}$$

- Denote

$$\begin{aligned} \|A\|_1 &= \sum_i \sigma_i(A) \\ \|A\|_\infty &= \sigma_1(A) \\ \|A\|_2 &= \sqrt{\sum_i \sigma_i^2(A)} \end{aligned}$$

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$$\|A\|_1 = \sum_i \sigma_i(A) \quad \text{“Trace Norm”}$$

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Trace norm method

Replace the rank objective function by the trace norm.

As rank equals the number of nonzero singular values, we have

$$\sum_i \sigma_i(A) \leq \sqrt{\operatorname{rank}(A)}\ \|A\|_2$$

For a M-by-N sign matrix this gives

$$\operatorname{rank}(A) \geq \frac{\|A\|_1^2}{MN}$$

Example: Trace Norm

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Thus trace norm method gives a bound on the rank of

$$\frac{\|H\|_1^2}{\|H\|_2^2} = \frac{N^3}{N^2} = N$$

Trace norm method: drawback

Trace norm bound is not monotone. Consider

$$\begin{pmatrix} H_N & 1_N \\ 1_N & 1_N \end{pmatrix}$$

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Worse bound than on Hadamard submatrix!

A fix

We can fix this by considering

$$\max_{\substack{u,v \\ \|u\|=\|v\|=1}} \|A \circ uv^t\|_1$$

As this entrywise product does not increase the rank we still have

$$\mathrm{rank}(A) \geq \left(\frac{\|A \circ uv^t\|_1}{\|A \circ uv^t\|_2} \right)^2$$

For a sign matrix A, this simplifies nicely:

$$\mathrm{rank}(A) \geq \left(\|A \circ uv^t\|_1 \right)^2$$

We have arrived

At the γ_2 norm, introduced in communication complexity by [LMSS07, LS07].

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Has many nice properties. For this talk, we use the fact that it can be written as a semidefinite program and computed to high accuracy in polynomial time.

Application to approximation rank

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Use γ_2 norm as surrogate in rank minimization problem

$$\begin{aligned} \gamma_2^{\epsilon}(A) &= \min_B \gamma_2(B) \\ & |A(i,j) - B(i,j)| \leq \epsilon \text{ for all } i,j. \end{aligned}$$

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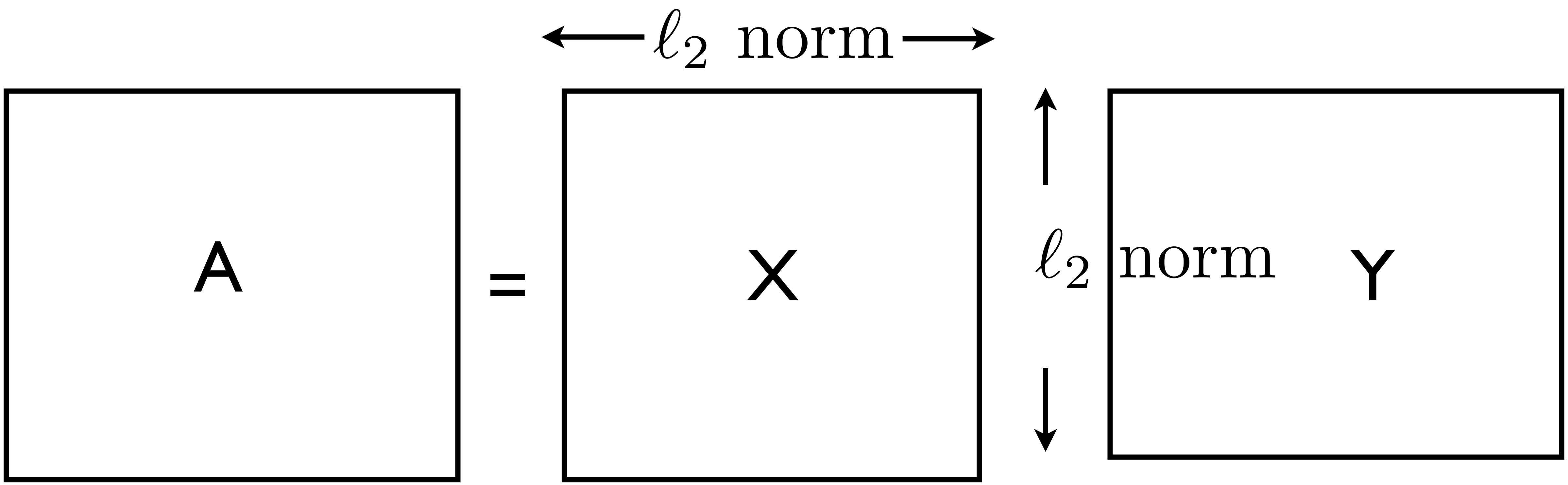
As argued above,

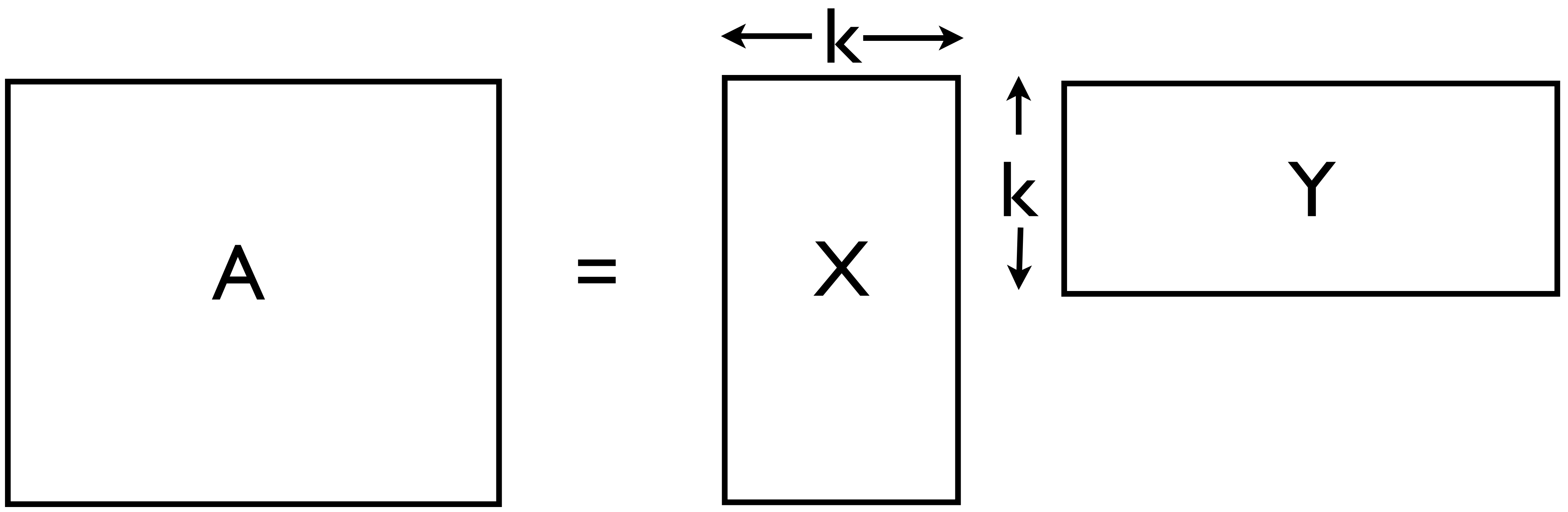
$$\frac{\gamma_2^\epsilon(A)^2}{(1+\epsilon)^2} \leq \mathrm{rank}_\epsilon(A).$$

γ_2 norm as factorization

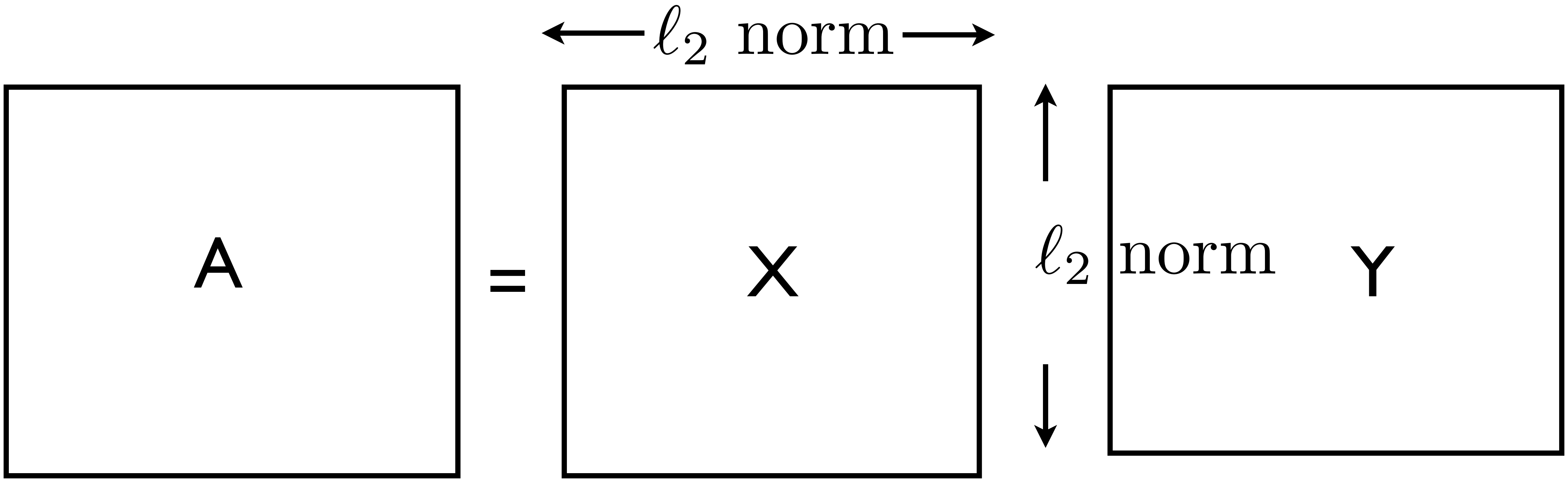


$$\gamma_2(A) = \min_{\substack{X,Y \\ XY=A}} \|X\|_r \|Y\|_c$$

Rank as factorization



γ_2 norm as factorization

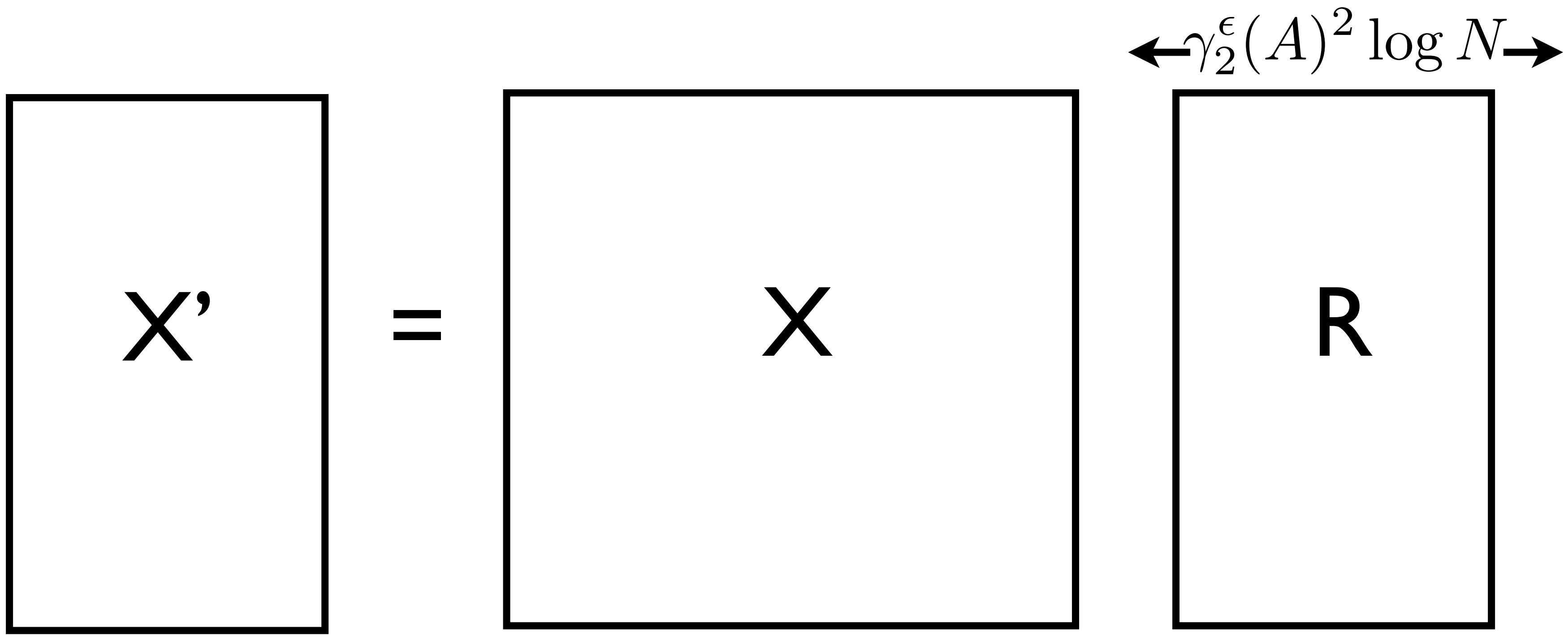


$$\gamma_2(A) = \min_{\substack{X,Y \\ XY=A}} \|X\|_r \|Y\|_c$$

Now shrink the rows

Take X,Y optimal factorization realizing $\gamma_2^{\epsilon}(A)$.

Obtain matrices X',Y' by randomly projecting columnspace down to dimension about $\gamma_2^{\epsilon}(A)^2 \log N$



Johnson-Lindenstrauss Lemma

Now X'Y' will be a matrix of the desired rank.

Furthermore, by the Johnson-Lindenstrauss lemma and a union bound, the inner products between all rows of X' and columns of Y' will approximately equal those between X and Y.

Thus X'Y' will still be entrywise close to A.

Error-reduction

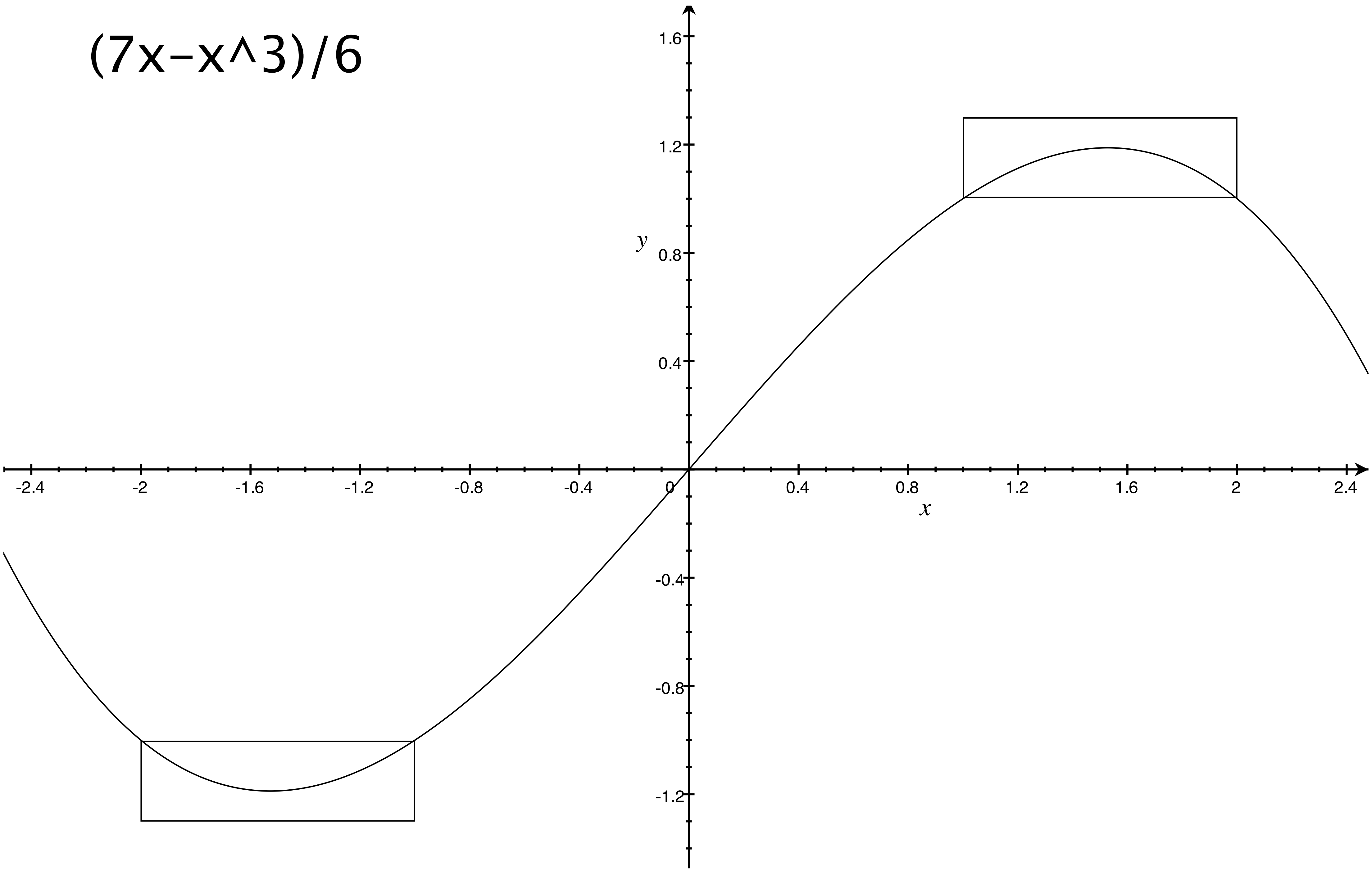
Can argue that if we started with an ε approximation, now will have 2ε approximation.

Can fix this by applying low degree polynomial approximation of the sign function entrywise to the matrix.

Applying a degree d polynomial blows up rank by at most a power of d.

Error-reduction

(7x−x^3)/6



Final result

For any M-by-N sign matrix A and constant $0 < \varepsilon < 1$

$$\frac{\gamma_2^\epsilon(A)^2}{(1+\epsilon)^2} \leq \mathrm{rank}_\epsilon(A) = O\left(\gamma_2^\epsilon(A)^2 \log(MN)\right)^3$$

Implies that the log approximation rank conjecture is essentially equivalent to the existence of a constant c such that

$$R_\epsilon(A) \leq (\log \gamma_2^\epsilon(A))^c + O(\log n).$$

Open questions

- What is the complexity of the real vector inner product function?
- Does the log rank conjecture imply the log approximation rank conjecture?
- Approximation algorithm for the limiting case of sign rank?