COMS W4203: Graph Theory - Final Exam Review

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Fundamentals

The final exam isn't "cumulative."

- Basic definitions (degree, isomorphism, girth, etc.)
- Eulerian/Hamiltonian conditions (even degree, Dirac)
- Menger (k-connected $\Leftrightarrow$ k disjoint paths), Kuratowski (planar $\Leftrightarrow$ no K_5 or $K_{3,3}$)

Fundamentals (cont.)

Problem

Let $d_1, \ldots d_n$ be a sequence of positive integers such that $\sum d_i = 2n-2$. Show that any such sequence is the degree sequence of a tree.

Problem

Consider the family of graphs $\mathcal{F}$ defined recursively as follows:

- $K_1 \in \mathcal{F}$.
- $G \in \mathcal{F} \Rightarrow \overline{G} \in \mathcal{F}$.
- $G, H \in \mathcal{F} \Rightarrow G \sqcup H \in \mathcal{F}$.

Show that every graph in $\mathcal{F}$ has no P_4 as an induced subgraph. Show that every complete multipartite graph is in $\mathcal{F}$.

Colorings

- Brooks ($\chi \leq \Delta$ except for K_n or C_n)
  - Pf. carefully choosing an ordering for greedy coloring.
- Konig (bipartite: $\chi' = \Delta$), Vizing ($\chi' \leq \Delta + 1$)
  - Pf. augmenting alternating paths.
- List coloring
- Chromatic polynomial

Colorings (cont.)

Problem

Let G be a k-critical graph. Show that if S is a separating set for G, then S does not induce a complete graph. Conclude that $|V(G)| \neq k+1$.

Problem

Let G be a graph on n vertices and m edges. Show that the chromatic polynomial $P_G(k)$ is of the form $k^n - mk^{n-1} \ldots$.

Flows/Circulations

- k-flows $\Leftrightarrow$ $\mathbb{Z}_k$-flows $\Leftrightarrow$ H-flows.
  - Pf. flip directions in the $\mathbb{Z}_k$-flow, flow polynomial.
- Various conditions for small flows.
- Seymour's 6-flow theorem.
  - Pf. find many even subgraphs and find a "3-flow" for the rest of the edges.
- Current graphs (a type of nowhere-zero flow?)

Flows/Circulations (cont.)

Problem

Find a 3-flow for ML_5.

Problem

Distribute the elements $1, \ldots, 6 \in \mathbb{Z}_{13}$ and give orientations to the edges of K_4 such that KCL holds in $\mathbb{Z}_{13}$. Why does this not give us a triangular embedding of K_{13} in some surface?

Extremal Graph Theory

- Turan (graph avoiding K_n with most edges is balanced complete $(n-1)$-partite)
  - Pf. "duplicating a vertex" to get a complete multipartite graph
- Ramsey ($R(s,t)$ is finite)
  - Pf. look at degree of any vertex and pigeonhole
- other subgraphs, minors, topological minors, etc.

Problem

Compute $R(C_4, C_4)$.

Random graphs

- Erdos-Renyi model $\mathcal{G}(n,p), \mathcal{G}(n,M)$
- Probabilistic method: union bound, linearity of expectation
  - $\Pr[X \vee Y] \leq \Pr[X] + \Pr[Y]$, $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$
- Approximating MAX-CUT (factor of $1/2$)
  - Pf. partition the vertices randomly
- Properties of almost all graphs
  - crucial that $n \to \infty$.

Problem

Is it true for all k that there is a tournament such that for every subset of k players, there's a player that beats all of them?

Random graphs

Problem

Is it true for all k that there is a tournament such that for every subset of k players, there's a player that beats all of them?

Proof.

Choose a random tournament on n vertices by assigning a random direction per edge with equal probability. Let I_S be the event that no vertex beats everyone in S. Then $\Pr[A_K] = (1 - 2^{-k})^{n-k}$, and

$$\Pr\left[\bigvee A_K\right] \leq \sum \Pr[A_k] = \binom{n}{k}(1 - 2^{-k})^{(n-k)}$$

which is less than 1 for sufficiently large n. $\square$

Embeddings on higher-order surfaces

- Classification of orientable surfaces, Euler's formula
  - $V - E + F = 2 - 2g$.
- Heawood inequality $\chi \leq (7 + \sqrt{1 + 48g})/2$
  - We saw a tight lower bound for $\chi = 12s + 7$.
- Genus under amalgamations.
  - additive w.r.t. bar amalgamations, vertex amalgamations, but not edge amalgamations
- Maximum genus (one-face embeddings)
  - Pf. find a splitting tree

Embeddings on higher-order surfaces

Problem

Let $\beta(G) = E - V + 1$ be the Betti number of G. Duke's conjecture states that if the genus of G is k, then $\beta(G) \geq 4k$.

- *Show there exists such a graph, i.e., demonstrating that this bound is best possible.*
- *Show that this is false for a cubic graph of girth 12.*

Embeddings on higher-order surfaces

Problem

- *Show that this is false for a cubic graph of girth 12.*

Proof.

For a cubic graph $\beta(G) = V/2 + 1$. By the edge-face inequality, we have $2E \geq 12F$, so substituting it into the Euler equation yields

$$2 - 2g = V - E + F \leq V - 5E/6 = -V/4,$$

Rearranging yields $V/2 + 1 \leq 4g - 3 < 4g$. $\square$