A sequence of integers is said to be *graphic* if it corresponds to the degree sequence of a simple graph. In these notes, the degree sequence is always sorted in descending order, and vertex v_i corresponds to the degree d_i.

Theorem 1 (Erdős-Gallai)**.** *A sequence of integers $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphic if and only if their sum is even and*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$$

for all $k = 1, \ldots, n$.

The requirement that the sum be even is just the handshaking lemma. The inequality may seem intimidating, but what it describes is an upper bound on the number of times an edge can be incident with one of the vertices with the k highest degrees. In particular, there are three parts:

- $\sum_{i=1}^{k} d_i$: the number of edge-vertex incidences with the first k vertices.
- $k(k-1)$: twice the number of edges in a complete graph on k vertices (i.e. if all the high-degree vertices were adjacent)
- $\sum_{i=k+1}^{n} \min(d_i, k)$: the summand is the maximum number of edges going from vertex v_i to the k vertices—you cannot exceed either the degree of v_i or k because we assumed the graph to be simple.

Proving that this condition is sufficient is tricky, and features some long computations and casework. Furthermore, this result does not give us a way to construct a graph realizing a given degree sequence. Personally, I prefer this alternative approach, which leads naturally to an algorithm:

Theorem 2 (Havel, Hakimi)**.** *Let $d_0 \geq d_1 \geq \cdots \geq d_{n-1}$ be a graphic sequence. Then there exists a graph with this degree sequence such that v_0 is adjacent to $v_1, v_2, \ldots, v_{d_0}$.*

Proof. Of all graphs with a given degree sequence, pick the one which maximizes $N(v_0) \cap \{v_1, \ldots, v_{d_0}\}$. If the intersection is exactly $N(v_0)$, then we are done. Otherwise, there exist $s \leq d_0$ such that v_0 and v_s are not neighbors, and $t > d_0$ such that v_0 and v_t are neighbors. Since we sorted the degrees, we know that $d_s \geq d_t$, and hence there is a vertex v_k adjacent to v_s but not v_t (one of the edges incident with v_t is used up on v_0). Adding the edges $v_0 v_s$, $v_t v_k$ and deleting $v_0 v_t$, $v_s v_k$ (see Figure 1) preserves all the degrees. It also increases the size of the intersection $N(v_0) \cap \{v_1, \ldots, v_{d_0}\}$, which is a contradiction. $\square$

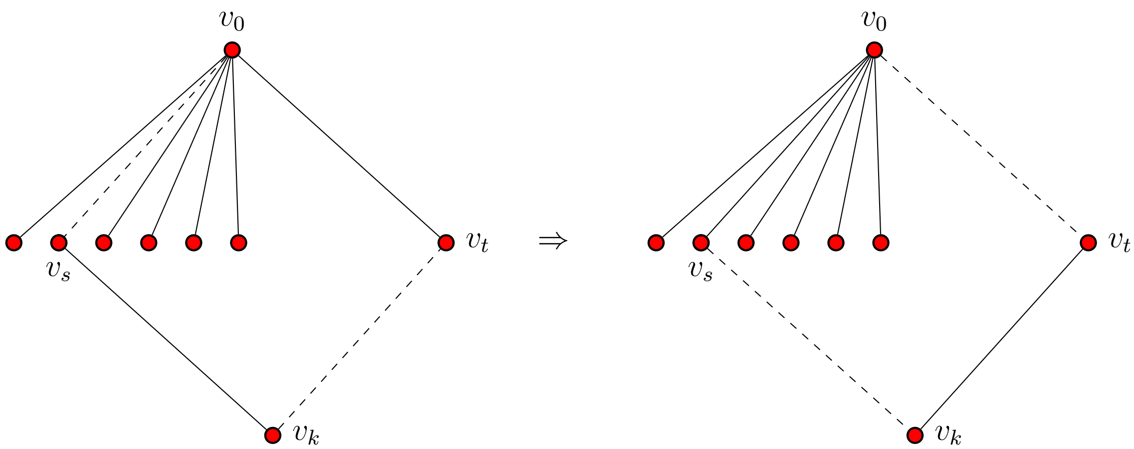

Figure 1: Exchanging edges while preserving the degree sequence.

To check if a sequence is graphic (this procedure can be modified to construct a graph), we

- delete v_0,
- decrement $d_1, \ldots, d_{d_0}$ by 1,
- recurse on $d_1, \ldots, d_{n-1}$ (after sorting the degrees)

If we end up with a degree sequence which is entirely 0, then the sequence is graphic.