loop is mentioned twice in the rotation at v. Let a "rotation system" on a graph be an assignment of a rotation to each vertex and a designation of orientation type for each edge. Then the preceding discussion can be summarized by the following theorem, used extensively by Ringel in the 1950s. The first formal proof was published by Stahl (1978).

Theorem 3.2.2. *Every rotation system on a graph G defines (up to equivalence of imbeddings) a unique locally oriented graph imbedding $G \to S$. Conversely, every locally oriented graph imbedding $G \to S$ defines a rotation system for G.* □

3.2.4. Pure Rotation Systems and Orientable Surfaces

Let the graph G have an imbedding in an oriented surface S. If the 0-bands of the associated band decomposition are given local orientations consistent with the orientation of the surface S, then every 1-band will be orientation-preserving. Conversely, any rotation system for the graph G such that every edge has type 0 induces an imbedding in an orientable surface (obtained by supplying the 2-bands) and a specific orientation of that surface. Theorem 3.2.3 summarizes this discussion. For the sake of brevity we define a "pure rotation system" for a graph to be one in which every edge has type 0.

Theorem 3.2.3. *Every pure rotation system for a graph G induces (up to orientation-preserving equivalence of imbeddings) a unique imbedding of G into an oriented surface. Conversely, every imbedding of a graph G into an oriented surface induces a unique pure rotation system for G.* □

Like Ringel (1974), we think it appropriate to ascribe credit for Theorem 3.2.3 jointly to Heffter (1891) and Edmonds (1960). See the Historical Note at the end of the section.

3.2.5. Drawings of Rotation Systems

There is a particularly simple way to incorporate into a drawing of a graph a rotation system for the graph: just be sure the clockwise order of the edges incident on a vertex in the drawing agrees with the assigned rotation at the vertex. The easiest way to do this is to draw first a dot for each vertex with spokes radiating from the dot labeled in clockwise order according to the rotation at the vertex. Then curves are drawn joining spokes with the same label. Finally, all type-1 edges are marked with a cross. The resulting drawing is called a "projection" of the given rotation. For instance, Figure 3.16 shows a projection of the pure rotation system defined by the toroidal imbedding of $K_{4,4}$ minus a 1-factor given in Figure 3.10. For this projection a judicious choice of the "under" edge at each crossing helps make the surface "visible".

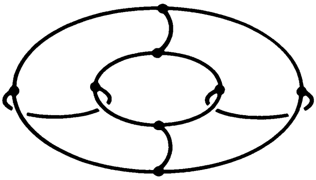

Figure 3.16. A rotation projection with no type-1 edges.

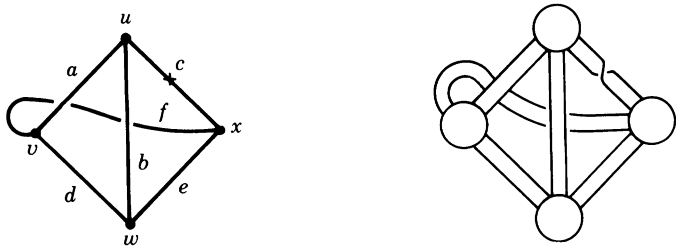

Figure 3.17. A rotation projection for K_4 and the corresponding reduced band decomposition.

If a given graph is simplicial, then the list format of a rotation system may give adjacent vertices instead of edges. This is called the "vertex form of a rotation system". In this context, it is also natural to list only the sequence of vertices in a boundary walk.

Example 3.2.4. *The rotation system for K_4 whose projection is illustrated on the left in Figure* 3.17 *can be given a list format in either the edge form or the vertex form.*

$u.$	$c^1\,b\,a$	$u.$	$x^1\,w\,v$
$v.$	$f\,a\,d$	$v.$	$x\,u\,w$
$w.$	$d\,b\,e$	$w.$	$v\,u\,x$
$x.$	$e\,f\,c^1$	$x.$	$w\,v\,u^1$

The corresponding reduced band decomposition is given at the right. By tracing along the boundary of this reduced band decomposition surface, one easily verifies that the imbedding has two faces $u\,v\,x\,w\,u\,x\,w\,v\,x$ and $u\,v\,w$ in vertex form, or $a\,f\,e\,b\,c\,e\,d\,f\,c$ and $a\,d\,b$ in edge form.

3.2.6. Tracing Faces

Given a rotation system for a graph, one frequently needs to obtain a listing or enumeration of the boundary walks of the reduced faces. If the rotation system is given by a projection, then the procedure is well illustrated in Example 3.2.4. First, thicken each edge into a 1-band, and give the band a twist if the edge is type 1. Then simply trace out, say with a pencil, the boundary components of the resulting surface. On the other hand, if the rotation system is given in list format, for example, as computer input, then this geometric method is impossible. Some thought about the geometric method should convince the reader that the following Face Tracing Algorithm is correct. We first introduce some helpful terminology. If the rotation at vertex v is $\ldots de \ldots$ (up to a cyclic permutation), then we say that d is "the edge before e at v", that e is "the edge after d at v", and that the edge pair (d, e) is a "corner at v with second edge e".

Face Tracing Algorithm. *Assume that the given graph G has no 2-valent vertices. Choose an initial vertex v_0 of G and a first edge e_1 incident on v_0. Let v_1 be the other endpoint of e_1. The second edge e_2 in the boundary walk is the edge after (resp., before) e_1 at v_1 if e_1 is type 0 (resp., type 1). If the edge e_1 is a loop, then e_2 is the edge after (resp., before) the other occurrence of e_1 at v_1. In general, if the walk traced so far ends with edge e_i at vertex v_i, then the next edge e_{i+1} is the edge after (resp., before) e_i at v_i if the walk so far is type 0 (resp., type 1). The boundary walk is finished at edge e_n if the next two edges in the walk would be e_1 and e_2 again. To start a different boundary walk, begin at the second edge of any corner that does not appear in any previously traced faces. If there are no unused corners, then all faces have been traced.*

Observe that the walk does not necessarily stop when the first edge e_1 is encountered a second time; we might not be on the same side of e_1 as at the beginning. The followup by the edge e_2 is what confirms that we are on the original side of e_1, assuming of course that the vertex v_1 does not have valence 2 (see Exercise 14).

Example 3.2.5. *Consider the rotation system*

$$u. \quad a^1\, f\, b\, d^1\, a^1\, e^1\, b\, c$$

$$v. \quad c\, f\, g$$

$$w. \quad e^1\, d^1\, g$$

Begin the first face at vertex u with the edge a at the corner (c, a). The next edge in the boundary walk is d, the edge before (since the walk a is type 1) the other occurrence of the edge a at the vertex u. The next edge is g, the edge after d (since the walk $a\,d$ is type 0) at the vertex w. The next edge is c. Since the following two edges would be a and d again, the walk terminates with c, yielding the face $a\,d\,g\,c$. Since the corner (a, f) has not yet appeared, we can begin a second boundary walk with edge f at vertex u. This time the face $f\,g\,e\,a$ is obtained. The third and fourth faces $b\,c\,f$ and $d\,e\,b$ are obtained in a similar fashion by starting with edge b from corner (f, b) and edge d from corner (b, d). Figure 3.18 shows these four faces, first separately, then assembled together.

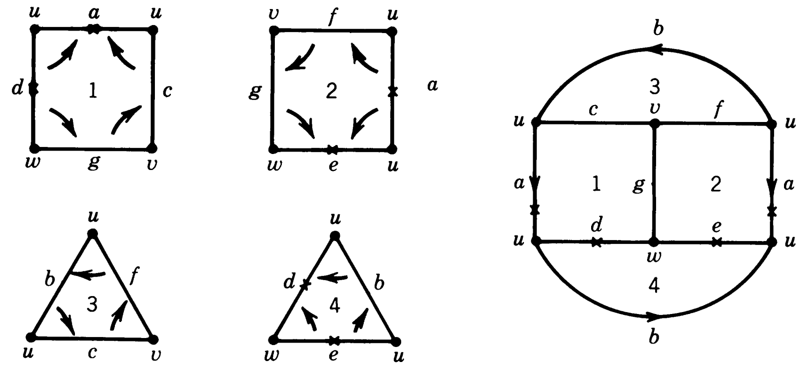

Figure 3.18. The faces from Example 3.2.5 assembled into a Klein bottle, so that the local orientations at vertices u, v, and w are correctly completed.

Example 3.2.6. *The rotation system*

$$\begin{aligned} t.&\quad d^1\, b\, e^1 \\ u.&\quad f\, d^1\, a^1 \\ v.&\quad c^1\, b\, a^1 \\ w.&\quad c^1\, e^1\, f \end{aligned}$$

has as its faces the polygons dfe, bad, ecb, and fca. When these faces are assembled, they yield a 2-sphere. Thus, as we have mentioned, the existence of orientation reversing edges need not mean that the surface is nonorientable, provided that no cycle contains an odd number of orientation reversing edges.

3.2.7. Duality

Let B be a band decomposition of the surface S. Suppose each 2-band has been given a specific orientation. The dual band decomposition of B, denoted B^*, is defined by letting the i-bands of B be the $(2-i)$-bands of B^*, for $i = 0, 1, 2$; the ends of each 1-band of B become the sides of the corresponding 1-band of B^* and vice versa. If $G \to S$ is the graph imbedding associated with the decomposition B, then the imbedding associated with B^* is naturally the dual imbedding $G^* \to S^*$.

Let e be an edge of the imbedding $G \to S$ associated with the decomposition B. The orientation type of the dual edge e^* depends on the choice of orientations for the 2-bands of the primal imbedding or, equivalently, on the direction given to the closed walk around the boundary of each primal face. The edge e appears twice in the course of listing all directed face boundaries. If the two appearances have opposite directions, then the dual edge e^* is type 0; otherwise it is type 1. Thus, in Example 3.2.4, if the directed boundaries are $uvxwuxwvx$ and uvw, then $(ux)^*$, $(vw)^*$, and $(wx)^*$ are type 0, whereas